

# Limit Domains in Several Complex Variables

By

Alexander Console

Submitted to the graduate degree program in Mathematics and the  
Graduate Faculty of the University of Kansas  
in partial fulfillment of the requirements for the degree of  
Doctor of Philosophy

Committee members

---

Estela Gavosto, Chairperson

---

James Orr

---

Jack Porter

---

Albert Sheu

---

Rodolfo Torres

Date defended: 

---

December 12, 2013

The Dissertation Committee for Alexander Console certifies  
that this is the approved version of the following dissertation :

Limit Domains in Several Complex Variables

---

Estela Gavosto, Chairperson

Date approved: December 12, 2013

# Abstract

In this thesis we demonstrated the existence of domains in  $\mathbb{C}^2$  evidencing both intrinsic phenomena of  $\mathbb{C}^n$ ,  $n \geq 2$ , and different types of boundary smoothness. We constructed these domains by taking limits of preimages of polydiscs under a sequence of shears selected to control boundary smoothness.

Unlike the complex plane, in  $\mathbb{C}^n$  there are simply connected domains that are biholomorphic to  $\mathbb{C}^n$  but are proper subsets of  $\mathbb{C}^n$ . These domains are called Fatou-Bieberbach domains and they arise naturally in the study of complex dynamics. We showed that there exists a Fatou-Bieberbach domain in  $\mathbb{C}^2$  with Gevrey smooth boundary.

Another interesting occurrence in  $\mathbb{C}^2$  is the existence of simply connected proper subsets of  $\mathbb{C}^2$  that are not biholomorphic to the unit ball nor biholomorphic to  $\mathbb{C}^2$ . One such class are Short- $\mathbb{C}^2$  domains. We constructed Short- $\mathbb{C}^2$  domains with  $C^\infty$  boundary and Short- $\mathbb{C}^2$  domains with prescribed local  $C^\ell$  boundary smoothness and controlled geometry.

## **Acknowledgements**

I would like to acknowledge my gratefulness to my family and friends for their support, to my advisor for years of help, encouragement, and steadfast confidence in me, to all my professors who taught me so much in my classwork at KU, and to my fellow graduate students, current and former, who helped me every step along the way.

To my mother and father.

# Contents

<b>1</b>	<b>Introduction</b>	<b>1</b>
<b>2</b>	<b>Fatou-Bieberbach Domains</b>	<b>5</b>
2.1	History and Definitions . . . . .	5
2.2	Rosay and Rudin Construction . . . . .	6
2.3	A Simple Fatou-Bieberbach Domain . . . . .	7
2.4	Boundary Smoothness of Fatou-Bieberbach Domains . . . . .	12
<b>3</b>	<b>Fatou-Bieberbach Domains with local <math>C^\ell</math>-Smooth Boundary</b>	<b>14</b>
3.1	Fatou-Bieberbach Domains Generated by Varying Sequences of Automorphisms .	14
3.2	Push-Out Constructions of Fatou-Bieberbach Domains . . . . .	15
3.3	Existence of Fatou-Bieberbach Domains with Locally $C^\ell$ -Smooth Boundary . . . .	19
3.4	Push-Out Lemmas . . . . .	22
3.5	Showing Boundary Smoothness . . . . .	25
<b>4</b>	<b>Fatou-Bieberbach Domain with Gevrey Class Boundary</b>	<b>42</b>
4.1	Stensönes' Example . . . . .	42
4.2	Gevrey Class . . . . .	44
4.3	A Gevrey Class-Smooth Fatou-Bieberbach Domain . . . . .	45
<b>5</b>	<b>Short-<math>\mathbb{C}^2</math> Domains with Local <math>C^\ell</math>-Smooth Boundary</b>	<b>49</b>
5.1	Fornæss' Construction of Short- $\mathbb{C}^2$ Domains . . . . .	49

5.2	Short $\mathbb{C}^2$ with local $C^\ell$ -boundary . . . . .	52
5.3	Short $\mathbb{C}^2$ as a Limit of Polydisc $\Omega_n$ 's . . . . .	54
5.4	The Smoothness Argument . . . . .	59
5.5	Two Smoothness Lemmas . . . . .	60
<b>6</b>	<b>Short-<math>\mathbb{C}^2</math> with <math>C^\infty</math>-Smooth Boundary</b>	<b>65</b>
6.1	Proof of the Existence of a Short- $\mathbb{C}^2$ Domain $\Omega$ with $C^\infty$ -Smooth Boundary . . . .	70
<b>7</b>	<b>Future Work</b>	<b>74</b>

# Chapter 1

## Introduction

In the theory of functions of a single complex variable, the simply connected open subsets of the complex plane are completely characterized by the Riemann Mapping Theorem. In particular, the automorphism group for the complex plane is given by linear maps  $z \mapsto az + b$ . In the theory of functions of several complex variables, there is no theorem equivalent to the Riemann Mapping Theorem, and the automorphism groups are much more varied. In particular, if  $n \geq 2$ , the space  $\mathbb{C}^n$  has a rich group of automorphisms. For example, if we pick any holomorphic function  $f : \mathbb{C} \rightarrow \mathbb{C}$ , the map  $(z, w) \mapsto (z, w + f(z))$  is an automorphism of  $\mathbb{C}^2$ .

To clarify the situation in the complex plane we adopt the following definitions.

**Definitions 1.** *A map  $f : \mathbb{C} \rightarrow \mathbb{C}$  is called holomorphic if it is complex differentiable.*

*Two nonempty open subsets  $U, V$  of  $\mathbb{C}$  are biholomorphic to one another or biholomorphically equivalent if there is a holomorphic map  $F : U \rightarrow V$  mapping  $U$  onto  $V$  with holomorphic inverse  $F^{-1} : V \rightarrow U$  mapping  $V$  onto  $U$ . The map  $F$  is called a biholomorphism. A biholomorphic map  $F : \mathbb{C} \rightarrow \mathbb{C}$  with image  $F(\mathbb{C}) = \mathbb{C}$  is called an automorphism of  $\mathbb{C}$ .*

As mentioned above the Riemann Mapping Theorem determines the biholomorphic equivalence of domains in the complex plane.

**Theorem 1** (The Riemann Mapping Theorem). *Every non-empty, simply connected, open proper subset of  $\mathbb{C}$  is biholomorphic to the open unit disc.*



There are no proper sub-domains  $\Omega \subset \mathbb{C}^n$  that are biholomorphic to  $\mathbb{C}^n$  for  $n = 1$ . This fact follows from the Riemann Mapping Theorem and Louisville's Theorem. Recall Louisville's Theorem:

**Theorem 2** (Louisville's Theorem). *The only bounded holomorphic functions  $f : \mathbb{C} \rightarrow \mathbb{C}$  are the constant functions.*

Now note that being simply connected is a topological property invariant under homeomorphism, and hence under biholomorphism. Since  $\mathbb{C}$  itself is simply connected, any domain biholomorphic to  $\mathbb{C}$  must be simply connected. Assume there is a domain  $\Omega$  in  $\mathbb{C}$ , such that  $\Omega$  is both a proper subset of  $\mathbb{C}$  and  $\Omega$  is biholomorphic to  $\mathbb{C}$ . By the Riemann Mapping Theorem, this domain  $\Omega$  would be biholomorphic to the unit disc in  $\mathbb{C}$ . The unit disc is bounded in modulus, so by Louisville's Theorem the biholomorphism is constant, which is a contradiction.

We shall now turn our attention to higher dimensions, i.e.  $\mathbb{C}^n, n \geq 2$ .

**Definitions 2.** *A map  $f : \mathbb{C}^n \rightarrow \mathbb{C}^n$ , is called holomorphic if  $f(z_1, z_2, \dots, z_n)$  is holomorphic in each variable separately.*

*Two nonempty open subsets  $U, V$  of  $\mathbb{C}^n$  are biholomorphic to one another or biholomorphically equivalent if there is a holomorphic map  $F : U \rightarrow V$  mapping  $U$  onto  $V$  with holomorphic inverse  $F^{-1} : V \rightarrow U$  mapping  $V$  onto  $U$ . The map  $F$  is called a biholomorphism.*

*A biholomorphic map  $F : \mathbb{C}^n \rightarrow \mathbb{C}^n$  with image  $F(\mathbb{C}^n) = \mathbb{C}^n$  is called an automorphism of  $\mathbb{C}^n$ .*

One of the fundamental differences between the theory of functions of one complex variable and the theory of functions of several complex variables is the fact that the Riemann Mapping Theorem does not extend to higher dimensions. For example, we note that Poincaré in 1907 showed that the unit ball and the unit polydisc are *not* biholomorphic in  $\mathbb{C}^n, n > 1$  [Poincaré, 1907]. Burns and Shnider in 1975 showed that small  $C^\infty$  perturbations of the unit ball in  $\mathbb{C}^2$  are not biholomorphic to the unit ball. [Burns & Shnider, 1975]. Given these results, we may ask: For  $n \geq 2$ , can there exist domains  $\Omega \subsetneq \mathbb{C}^n$ , such that  $\Omega$  is biholomorphic to  $\mathbb{C}^n$ ? The answer to this question is yes.

Study of such domains began in the 1920's with the work of P. Fatou and L. Bieberbach, and the domains are called Fatou-Bieberbach domains. See, for example [Fatou, 1922] and [Bieberbach, 1933]. However, Dixon and Esterle in [Dixon & Esterle, 1986] note that Poincaré observed the existence of non-degenerate (nonzero Jacobian) holomorphic functions on  $\mathbb{C}^2$  with non-dense range more than thirty years earlier in [Poincaré, 1890, pg. 333]. These early Fatou-Bieberbach domains were constructed as basins of attraction of fixed sequences of polynomial automorphisms. This method tends to produce Fatou-Bieberbach domains with fractal-like boundary. We will use a different method, with a varying sequence of polynomial automorphisms, to construct some of these domains in a way that will give them some boundary smoothness.

Later we will choose other sequences of polynomial automorphisms to construct related domains called “short- $\mathbb{C}^2$ ” domains. Whereas the sequences of polynomial automorphisms we use to generate Fatou-Bieberbach domains will generally vary greatly in both coefficients and degree, for the short- $\mathbb{C}^2$  domains we restrict the degree of the polynomial automorphisms. We will construct the polynomial automorphisms so that the resulting short- $\mathbb{C}^2$  domains have some boundary smoothness. In each case, with the sequence of automorphisms we will associate a sequence of domains, and the limiting domains generated by these sequences will have some boundary smoothness.

More specifically, in this thesis our primary concern will be the boundary smoothness of domains  $\Omega \subset \mathbb{C}^2$  that arise from taking a union of approximating domains  $\{\Omega_n\}$ , where:

- 1)  $\Omega_n \subsetneq \mathbb{C}^2$ , for every  $n \in \mathbb{N}$ .
- 2) Each  $\Omega_n$  is biholomorphic to the unit polydisc.
- 3)  $\Omega_1 \subset \Omega_2 \subset \cdots \subset \bigcup_{j=1}^{\infty} \Omega_j = \Omega$ .

An increasing union of simply connected domains in  $\Omega_n \subset \mathbb{C}$  is simply connected, and therefore the limiting procedure above, when applied to domains  $\Omega_n \subset \mathbb{C}$ , either results in  $\mathbb{C}$  or a domain biholomorphic to the unit disc. Our choices of  $\Omega_n$  will union up to domains that are neither  $\mathbb{C}^2$ , nor are they biholomorphic to the polydisc (nor the unit ball).

Generally our proofs will use a limit process to generate the limit domain  $\Omega$ . We will prove lemmas that allow us to choose the first approximating domain  $\Omega_1$  with boundary smoothness in a polydisc about the origin. Then our limit process will supply a sequence of approximating domains that will preserve or increase boundary smoothness.

# Chapter 2

## Fatou-Bieberbach Domains

### 2.1 History and Definitions

Let  $n \in \mathbb{N}$  be greater than or equal to two. A Fatou-Bieberbach domain  $\Omega$  is a nonempty open proper subset of  $\mathbb{C}^n$  that is biholomorphic to  $\mathbb{C}^n$ . Recall that there are no Fatou-Bieberbach domains in  $\mathbb{C}^1$  by the Riemann Mapping Theorem. Fatou-Bieberbach domains were first described as basins of attraction, which we now define.

**Definitions 3.** Let  $F : \mathbb{C}^2 \rightarrow \mathbb{C}^2$  be biholomorphic. Let  $F^k = \overbrace{F \circ F \circ \cdots \circ F}^{k \text{ } F\text{'s}}$  denote the  $k$ -th iterate of  $F$ . If  $p \in \mathbb{C}^2$  and  $F(p) = p$  then  $p$  is said to be a fixed point of  $F$ . If  $p$  is a fixed point of  $F$  and the eigenvalues  $\lambda_1, \lambda_2$  of  $F'(p)$ , the Jacobian of  $F$  at  $p$ , satisfy the condition  $|\lambda_1| < 1, |\lambda_2| < 1$ , then  $p$  is said to be attracting. If  $p$  is an attracting fixed point of  $F$ , the set  $\Omega_p = \{z \in \mathbb{C}^2 : \lim_{k \rightarrow \infty} F^k(z) = p\}$  is called the basin or basin of attraction of  $p$ .

In the context of our theme of limiting domains, suppose that  $p$  is an attracting fixed point of  $F$ . Generally we will assume  $p = 0$ . Since the basin of attraction consists of points whose successive images under  $F$  converge to  $p$ , we may pick a constant  $\varepsilon$  close to zero, say  $0 < \varepsilon < \frac{1}{2}$ , and for each  $n \in \mathbb{N}$  define the approximating domain  $\Omega_n$  to be the preimage of the set  $\{(z, w) \in \mathbb{C}^2 : |z| < \varepsilon, |w| < \varepsilon\}$  under the map  $F^k$ . Then the  $\Omega_n$ 's are nested, biholomorphic to the unit polydisc, and their limit is the basin of attraction at  $p$ .

The first Fatou-Bieberbach domains studied were constructed by iterating a fixed automorphism  $F : \mathbb{C}^2 \rightarrow \mathbb{C}^2$  which generated the domain as the basin of attraction at a fixed point. Arguably the first such Fatou-Bieberbach Domain construction was due to Fatou in [Fatou, 1922]. Fatou's work has historical significance and also allows us to efficiently illustrate the main difficulties involved in constructing a Fatou-Bieberbach Domain as a basin of attraction. Unfortunately while Fatou's method is generally workable, his example was not correct because he chose to iterate the function  $F : (z, w) \mapsto (w, 2z + 4zw - 3w^2)$ , and this function is not injective. For example, in [Dixon & Esterle, 1986] Dixon and Esterle observed that  $F(z, -\frac{1}{2}) = (-\frac{1}{2}, -\frac{3}{4})$  for any  $z \in \mathbb{C}$ .

Later on, Bieberbach in [Bieberbach, 1933] provided the example  $F(z, w) \mapsto (4z + 2w^5 - 5w^2, 4w + 2z^5 - 5z^2)$ . While Bieberbach's map is a genuine automorphism, it is somewhat complicated. Therefore we present a simpler alternative to Fatou's domain due to B.V. Shabat in [Shabat, 1992].

The following examples illustrate some features of such domains. We will consider, in turn, Rosay and Rudin's method of theorem on constructing Fatou-Bieberbach domains, B.V. Shabat's version of Fatou's Domain, and some results of Bedford and Smillie and results of Fornaess and Sibony that illustrate the pathological boundary behavior that can arise in these domains.

## 2.2 Rosay and Rudin Construction

In their seminal paper [Rosay & Rudin, 1988], Rosay and Rudin proved the following theorem which allows us to generate a Fatou-Bieberbach domain as a basin of attraction of an automorphism, without the laborious methods of Fatou and Beiberbach:

**Theorem 3.** [Rosay & Rudin, 1988] *Suppose that  $F$  is an automorphism of  $\mathbb{C}^n$ ,  $F(0) = 0$ , and all the eigenvalues  $\lambda_i$  of  $F'(0)$  satisfy  $|\lambda_i| < 1$ . Then there exists a biholomorphic map  $\Phi$  from  $\mathbb{C}^n$  onto the region*

$$\Omega = \left\{ z \in \mathbb{C}^n : \lim_{k \rightarrow \infty} F^k(z) = 0 \right\}.$$

*Moreover,  $\Phi$  can be chosen so that  $J\Phi$  is identically one if  $JF$  is constant.*

### Outline of the proof of theorem 3:

The idea of the proof is to find an automorphism  $G$  whose basin of attraction at the origin is all of  $\mathbb{C}^n$  and to show that this basin is biholomorphic to the basin of attraction of  $F$  at the origin. Suppose that  $F$  and  $G$  are two automorphisms of  $\mathbb{C}^n$  and  $p$  is a point in  $\mathbb{C}^n$ . We say that  $F$  and  $G$  are "locally conjugate at the point  $p$ " if there exists a function  $\phi$ , holomorphic in a neighborhood of  $p$ , such that  $F = \phi^{-1} \circ G \circ \phi$ . Roughly, the proof has two steps, namely:

- 1) Firstly one proves that if  $F, G$  are automorphisms of  $\mathbb{C}^n$  which: fix the point  $p$ , are attracting at  $p$ , and  $F, G$  are locally conjugate at  $p$ , then the basins of attraction for  $F$  and  $G$  at  $p$  are biholomorphic.
- 2) Then one shows that  $F$  is locally conjugate at  $p$  to an automorphism whose attracting basin at  $p$  is  $\mathbb{C}^n$ . In particular, one shows that  $F$  is locally conjugate to a "lower triangular polynomial automorphism", i.e. a map of the form

$$G(z) = A(z) + (0, g_2(z), \dots, g_n(z))$$

where  $A$  is a lower triangular matrix whose eigenvalues are the same as  $F'(p)$  and each  $g_j(z)$  is a polynomial of degree greater than one in the variables  $z_1, \dots, z_{j-1}$ .

For the details, the interested reader should consult [Rosay & Rudin, 1988]. The following example illustrates the main ideas of the proof of theorem 3.

## 2.3 A Simple Fatou-Bieberbach Domain

**Theorem 4.** [Shabat, 1992] Let  $0 < a < 1$  and define the automorphism  $\eta$  of  $\mathbb{C}^2$  by

$$\eta : (z_1, z_2) \mapsto (z_2, a^2 z_1 + (1 - a^2) z_2^2).$$

Then the basin of attraction of the origin

$$\Omega = \left\{ z \in \mathbb{C}^2 : \lim_{k \rightarrow \infty} \eta^k(z) = 0 \right\}$$

is a Fatou-Bieberbach domain.

We note that in the modified Fatou example presented above,  $F = \eta$  and because  $F$  has two fixed points, by theorem 3 the basin at the origin is a Fatou-Bieberbach Domain.

Roughly, the idea of the proof is to show that there exist a function  $\Phi$ , holomorphic in a neighborhood of the origin in  $\mathbb{C}^2$ , and a linear automorphism  $A$  of  $\mathbb{C}^2$ , where the basin of attraction of  $A$  at the origin is all of  $\mathbb{C}^2$ . Then we show  $\Phi$  satisfies the equation  $\Phi = \eta^{-1} \circ \Phi \circ A$ . In other words, taking the liberty of denoting the restriction of  $\eta$  to  $\Omega$  by  $\eta$ , this diagram:

$$\begin{array}{ccc} \mathbb{C}^2 & \xrightarrow{A} & \mathbb{C}^2 \\ \phi \downarrow & & \downarrow \phi \\ \Omega & \xrightarrow{\eta} & \Omega \end{array}$$

is commutative. It follows that for any positive integer  $k$ ,  $\Phi = \eta^{-k} \circ \Phi \circ A^k$ . Now we see that as  $k \rightarrow \infty$ ,  $A^k$  sends all of  $\mathbb{C}^2$  to the origin,  $\Phi$  fixes the origin, but then iterating  $\eta^{-k}$ , with  $k \rightarrow \infty$ , will yield the basin of attraction of  $\eta$  at the origin, which is  $\Omega$ . So  $\Phi$  maps  $\mathbb{C}^2$  biholomorphically onto  $\Omega$ .

The automorphism

$$\eta(z_1, z_2) = (z_2, a^2 z_1 + (1 - a^2) z_2^2)$$

has two fixed points,  $(1, 1)$  and  $(0, 0)$ .

$$D\eta(z_1, z_2) = \begin{pmatrix} 0 & 1 \\ a^2 & 2z_2(1 - a^2) \end{pmatrix}$$

The eigenvalues of  $D\eta$  at  $(0,0)$  are  $\pm a$ , so  $(0,0)$  is an attracting point. Define the automorphism

$$A(z_1, z_2) = (-az_1, az_2).$$

It will be shown that a solution to the functional equation:

$$\eta \circ \Phi = \Phi \circ A$$

that is biholomorphic in a neighborhood of the origin is a biholomorphism  $\Phi : \Omega \rightarrow \mathbb{C}^2$ , with  $\Phi(\Omega) = \mathbb{C}^2$ . Let  $\Phi = (\Phi_1, \Phi_2)$ . Then the functional equation  $\eta \circ \Phi = \Phi \circ A$  gives

$$\eta(\Phi_1, \Phi_2) = (\Phi_2, a^2\Phi_1 + (1-a^2)\Phi_2^2) = (\Phi_1(A), \Phi_2(A)) = \Phi(A).$$

$$\Phi_1(Az) = \Phi_2(z) \tag{2.3.1}$$

$$\Phi_2(Az) = a^2\Phi_1(z) + (1-a^2)\Phi_2^2(z) \tag{2.3.2}$$

Substituting (1) into (2) yields an equation in  $\Phi_1$ :

$$\Phi_1(A^2) = a^2\Phi_1 + (1-a^2)\Phi_1^2(A). \tag{2.3.3}$$

We will show that a solution of the functional equation exists, is holomorphic in a neighborhood of the origin, and is of the form:

$$\Phi_1(z) = z_1 + z_2 + \sum_{|k| \geq 2} b_k z^k = z_1 + z_2 + \psi(z), \quad k = (k_1, k_2), z = (z_1, z_2).$$

Notice that such a  $\psi$  satisfies the equation

$$a^2\psi - \psi(A^2z) = (a^2 - 1)(a(z_2 - z_1) + \psi(Az))^2. \tag{2.3.4}$$



Define  $\|\sum a_k z^k\| = \sum |a_k| z^k$  and define  $\sum a_k z^k \succeq \sum b_k z^k$  to mean  $|a_k| \geq |b_k|$  for all  $k$ .

Since

$$A^2 z = (a^2 z_1, a^2 z_2)$$

$$a^2 \psi(z) - \psi(A^2 z) = \sum_{|k| \geq 2} b_k (a^2 - a^{2|k|}) z^k$$

$$a^2 (1 - a^2) \|\psi(z)\| \preceq \|a^2 \psi(z) - \psi(A^2 z)\|.$$

Similarly,

$$\left\| (a(z_2 - z_1) + \psi(Az))^2 \right\| \preceq a^2 (z_1 + z_2 + a \|\psi(z)\|)^2$$

Thus

$$\|\psi(z)\| \preceq (z_1 + z_2 + a \|\psi(z)\|)^2. \quad (2.3.5)$$

Note that the equation  $\theta(t) = (t + a\theta(t))^2$  has solution

$$\theta(t) = \frac{1}{2a^2} \left( 1 - 2at - \sqrt{1 - 4at} \right) = \sum_{j \geq 2} c_j t^j.$$

where  $c_j > 0$  for  $j \geq 2$  and this series converges in a neighborhood of the origin. Inducting on  $|k|$  we can show that

$$\psi(z) \preceq \theta(z_1 + z_2).$$

Firstly note that

$$\begin{aligned} \theta'(t) &= \frac{1}{a} \left( (1 - 4at)^{-\frac{1}{2}} - 1 \right) & \theta'(0) &= 0 \\ \theta''(t) &= 2(1 - 4at)^{-\frac{3}{2}} & \theta''(0) &= 2 \end{aligned} \quad \text{Since (2.3.5) shows that the second degree terms}$$

of  $\Phi$  are  $\preceq$ -dominated by  $z_1^2 + z_2^2$ , the case  $n = 2$  in holds. Now assume that for  $k = 1, \dots, n$ ,  $|b_k| \leq c_k$ . Looking at the expression

$$(z_1 + z_2 + a \|\psi(z)\|)^2$$

we see that the coefficients of the  $(n+1)$ -degree monic terms are either of the form  $b_\eta$  where  $|\eta| = n$  or of the form  $b_\gamma b_\nu$  where  $|\gamma| \geq 2, |\nu| \geq 2$ , and  $|\gamma + \nu| = n+1$ . In the first case, by the inductive assumption these  $b_\eta$  are dominated by  $c_n$  and in the second case  $b_\gamma b_\nu$  is dominated by  $c_{|\gamma|} c_{|\nu|}$ . Then using (2.3.5), notice

$$\psi(z) \preceq (z_1 + z_2 + a \|\psi(z)\|)^2 \preceq (z_1 + z_2 + a\theta(z_1 + z_2))^2 = \theta(z_1 + z_2).$$

So by comparison  $\psi(z)$  converges in a neighborhood of the origin. Taking the identity:  
 $\Phi = \eta^{-1} \circ \Phi \circ A$  and iterating gives:

$$\Phi = \eta^{-1} \circ \Phi \circ A = \eta^{-1} \circ (\eta^{-1} \circ \Phi \circ A) \circ A$$

and hence

$$\Phi = \eta^{-k} \circ \Phi \circ A^k \text{ for } k > 0.$$

The Jacobian determinant of  $A$  is

$$J_A = -a^2 = J_\eta, \quad a^2 < 1.$$

Therefore  $A$  is contractible. Applying the chain rule to the functional equation,  $\eta \circ \Phi = \Phi \circ A$ :

$$J_\eta(\Phi) J_\Phi = J_\Phi(A) J_A$$

$$(-a^2) J_\Phi = J_\Phi(A) (-a^2)$$

$$J_\Phi = J_\Phi(A).$$

Iterating yields  $J_\Phi = J_\Phi(A^k)$ . Since  $A^k \rightarrow 0$  as  $k \rightarrow \infty$  and  $J_\Phi$  is continuous,  $J_\Phi = J_\Phi(0)$ .

Expanding  $\Phi$

$$\Phi(z_1, z_2) = \begin{cases} \Phi_1(z_1, z_2) = z_1 + z_2 + \text{hot} \\ \Phi_2(z_1, z_2) = -az_1 + az_2 + \text{hot} \end{cases}$$

(by the abbreviation *hot* we mean "higher order terms".) So  $J_\Phi(0) = 2a \neq 0$ . Then  $\Phi$  is biholomorphic in a neighborhood of the origin and hence on  $\mathbb{C}^2$  by the identity  $\Phi = \eta^{-k} \circ \Phi \circ A^k$ .

Using the identities  $\eta^{-k} \circ \Phi = \Phi \circ A^k$ , we will prove the equality:

$$\Omega = \{z : \lim_{k \rightarrow \infty} \eta^k(z) = 0\} = \Phi(\mathbb{C}^2).$$

Let  $z \in \Omega$ , i.e.  $\lim_{k \rightarrow \infty} \eta^k(z) = 0$ . Then for large enough  $k$  there is a  $\xi \in \mathbb{C}^2$  such that  $\eta^k(z) = \Phi(\xi)$ .

The functional equation  $\eta^{-k} \circ \Phi = \Phi \circ A^k$  shows that

$$z = \eta^{-k}(\Phi(\xi)) = \Phi(A^k(\xi))$$

so  $z \in \Phi(\mathbb{C}^2)$ .

On the other hand, assume  $z \in \Phi(\mathbb{C}^2)$ . Then  $z = \Phi(\xi)$  for some  $\xi \in \mathbb{C}^2$ . The functional equation  $\eta^{-k} \circ \Phi = \Phi \circ A^k$  shows  $\eta^k(z) = \Phi(A^k \xi)$ . Taking the limit as  $k \rightarrow \infty$ ,  $\lim_{k \rightarrow \infty} \eta^k(z) = \Phi(0) = 0$ . Recall that  $\eta$  has two fixed points, so  $\Omega$  cannot be all of  $\mathbb{C}^2$ . Therefore  $\Omega$  is a Fatou-Bieberbach domain.

## 2.4 Boundary Smoothness of Fatou-Bieberbach Domains

### Boundaries of Basins of Attraction of Polynomial Automorphisms

Bedford and Smillie in [Bedford & Smillie, 1991] showed that members of the family of maps of the form  $g : \mathbb{C}^2 \rightarrow \mathbb{C}^2$ ,  $g : (z, w) \mapsto (z^2 + c + aw, az)$  for appropriate values of the parameters  $a, c$  have multiple basins of attraction, each of which is a Fatou-Bieberbach domain with common

boundary.

Fornæss and Sibony in [Fornæss & Sibony, 1991] proved that any basins of attraction  $\Omega_1, \Omega_2, \dots$  of a polynomial automorphism of  $\mathbb{C}^2$  share a common boundary, i.e.  $\partial\Omega_1 = \partial\Omega_2 = \dots$ . If the polynomial automorphism has two or more basins, the boundary cannot be a (topological) 3-manifold.

Given these pathological examples, we may ask, can Fatou-Bieberbach domains have smooth boundary? In the next chapter we will see that the answer to this question is yes.

# Chapter 3

## Fatou-Bieberbach Domains with local $C^\ell$ -Smooth Boundary

### 3.1 Fatou-Bieberbach Domains Generated by Varying Sequences of Automorphisms

In the previous chapter, we considered Fatou-Bieberbach domains that arose as the basin of attraction under iteration of a fixed automorphism  $F$ , that is,  $\Omega = \{(z, w) \in \mathbb{C}^2 : \lim_{n \rightarrow \infty} F^n(z, w) = 0\}$ . In this chapter we will consider Fatou-Bieberbach Domains constructed with a varying sequence of automorphisms, in a somewhat more direct fashion. Instead of generating  $\Omega$  as a basin of attraction we will compose a varying sequence of automorphisms  $\{G_n\}$  and in the limit the resulting biholomorphism will map  $\Omega$  onto  $\mathbb{C}^2$ . That is,

$$G = \lim_{n \rightarrow \infty} G_n \circ G_{n-1} \circ \cdots \circ G_1$$

$$G : \Omega \rightarrow \mathbb{C}^2$$

Our chief concern in each construction will be choosing the sequence of automorphisms so that we can control the boundary smoothness of the resulting Fatou-Bieberbach domain.

*Remark 1.* Using a manifold called "the abstract basin of attraction of a sequence of automorphisms" in [Fornæss & Stensønes, 2004], both the approaches to constructing Fatou-Bieberbach domains discussed above can be seen as constructing basins of attraction. We will not pursue this viewpoint.

## 3.2 Push-Out Constructions of Fatou-Bieberbach Domains

The push-out method was developed in previous work of Globevnik and Stensønes. In [Globevnik, 1997], [Forstnerič et al., 1996], [Globevnik & Stensønes, 1995], and in [Stensønes, 1997], where Stensønes constructed a Fatou-Bieberbach domain in  $\mathbb{C}^2$  with  $C^\infty$ -smooth boundary. The push-out method was developed to construct a Fatou-Bieberbach domains with certain properties. The idea is to construct approximating domains with the desired properties and then take a limit of these approximating domains such that the limit preserves these desired properties and is a Fatou-Bieberbach Domain  $\Omega$ . The name "push-out" comes from the fact that the domain  $\Omega$  is an increasing union of the approximating domains, where some set is "pushed out" of each approximating domain, assuring  $\Omega$  is not all of  $\mathbb{C}^2$ . We will use the push-out method to construct Fatou-Bieberbach domains in later chapters. The heart of the method is the lemma below. The version presented here is due to Forstnerič [Forstnerič, 2012].

Roughly, the lemma says we can construct biholomorphisms  $G$  of  $\mathbb{C}^2$ ,  $G : \Omega \rightarrow \mathbb{C}^2$ , by composing automorphisms  $H_j$  that look like the identity on larger and larger subsets of  $\mathbb{C}^2$ . The key is that for each point in  $z \in \Omega$  there is a positive integer  $k$  such that the orbit of  $z$  can vary markedly from the identity only for some finite sequence  $H_1, H_2, \dots, H_k$  of automorphisms, after which the images  $H_{k+1}(H_k \circ \dots \circ H_1(z)), H_{k+2}(H_{k+1} \circ \dots \circ H_1(z)), \dots$  are all very close in modulus. In the terms of the lemma, if  $z \in \Omega_k$ , then only at most the first  $k$  automorphisms can really move  $z$ , i.e. only for  $1 \leq j \leq k$  can the distance between  $H_j(z)$  and  $z$  substantially increase, because  $G_{-k}(z) \in D_k$ , and  $H_{k+m}$  looks like the identity on  $D_k$  since  $D_{m+k} \supset D_k$ , for  $m \geq k$ . Explicitly the lemma says:

**Lemma 3.2.1.** [Forstnerič, 2012] *Given a sequence of compact sets  $\{K_j\}$  in  $\mathbb{C}^2$ , satisfying*

$$i) K_0 \subset K_1 \subset K_2 \subset \dots \subset \cup_{j=0}^{\infty} K_j = \mathbb{C}^2$$

$$ii) K_{j-1} \subset \text{Int}(K_j) \text{ for } j \in \mathbb{N}.$$

Assume that there is a sequence of positive real numbers  $\{\varepsilon_j\}$ , with  $\sum_{j=0}^{\infty} \varepsilon_j < \infty$ , and a sequence of holomorphic automorphisms  $\{H_j\}$  of  $\mathbb{C}^2$  such that:

$$iii) \text{dist}(K_{j-1}, \mathbb{C}^2 \setminus K_j) > \varepsilon_j, \quad \forall j$$

$$iv) |H_j(z) - z| < \varepsilon_j, \quad \forall z \in K_{j-1}, j \in \mathbb{N}$$

Then for  $G_n = H_n \circ H_{n-1} \circ \dots \circ H_1$  there exists a domain  $\Omega \subset \mathbb{C}^2$  such that  $G = \lim_{n \rightarrow \infty} G_n$  converges uniformly on compact sets in  $\Omega$ , and  $G : \Omega \rightarrow \mathbb{C}^2$  is a biholomorphism. Furthermore,  $\Omega = \cup_{n=1}^{\infty} G_n^{-1}(K_n)$ .

The push out method is an application of lemma 3.2.1. We could take the maps  $H_j$  to be the identity on  $\mathbb{C}^2$  for all  $j$ , and then the  $K_j$  to be balls of radius  $j$  centered about the origin. Then the lemma would give the map  $G$  as the identity on  $\mathbb{C}^2$ , and  $\Omega$  would be all of  $\mathbb{C}^2$ . We want to construct Fatou-Bieberbach domains, so the idea is to choose maps  $H_j$  that push points in  $\mathbb{C}^2 \setminus K_{j-1}$  (where  $H_j$  need not behave like the identity map) out towards infinity, so that when we invert the maps  $H_j$ , they pull points in  $\mathbb{C}^2 \setminus K_{j-1}$  in toward the origin, and hence the resulting domain  $\Omega$  is strictly smaller than  $\mathbb{C}^2$ . More explicitly, recall that  $\Omega = \cup_{n=1}^{\infty} G_n^{-1}(K_n)$ . So  $z \notin \Omega$  only if

$$z \notin G_n^{-1}(K_n) \text{ for every } n.$$

This means that

$$G_n(z) \notin K_n \text{ for every } n.$$

In this sense, the  $G_n$ s can be chosen so that they "push"  $z \in \mathbb{C}^2 \setminus \Omega$  outside of the  $K_n$ s, and hence outside  $\Omega$ .

*Remark 2.*

$$\{z \in \mathbb{C}^2 : \{\|G_n(z)\|\}_{n \in \mathbb{N}} \text{ is bounded}\} = \cup_{n=1}^{\infty} (G_n)^{-1}(K_n)$$

Suppose that the sequence  $\{G_k(z)\}_k$  is bounded in modulus. Then  $\{G_k(z)\}_k \in K_j$  for some  $j = j(z) \in \mathbb{N}$ , hence  $z \in (G_j)^{-1}(K_j) \subset \cup_{m=1}^{\infty} (G_m)^{-1}(K_m)$ . (We have taken the liberty of identifying the sequence  $\{G_k(z)\}_k$  with its image.) On the other hand, if  $z \in \cup_{m=1}^{\infty} (G_m)^{-1}(K_m)$ ,  $z$  is in the domain of  $G$ , so the sequence  $\{G_k(z)\}$  converges to the value  $G(z)$ , hence the sequence  $\{G_k(z)\}$  is bounded.

In later theorems we will use the following version of the push-out lemma, which is due to Globevnik in [Globevnik, 1998].

**Lemma 3.2.2.** [Globevnik, 1998] *Given a sequence of domains  $D_n \subset \mathbb{C}^2$ ,  $D_n \subset\subset D_{n+1}$  for  $n \in \mathbb{N}$ , and  $\cup_{n=1}^{\infty} D_n = \mathbb{C}^2$ , let  $H_n$  be a sequence of holomorphic automorphisms of  $\mathbb{C}^2$  and  $\varepsilon_n$  a decreasing sequence of positive numbers such that for each  $n \in \mathbb{N}$ :*

$$a) \ H_{n+1}(D_n) \subset\subset D_{n+1}$$

$$b) \ |H_{n+1}(z) - z| < \frac{\varepsilon_n}{2^n} \quad z \in D_n$$

c) *if  $G : D_n \rightarrow \mathbb{C}^2$  is a holomorphic map such that  $|G(z) - z| < \varepsilon_n$  for  $z \in D_n$ , then  $G$  is injective on  $H_n(D_{n-1})$*

*Let  $G_n = H_n \circ H_{n-1} \circ \cdots \circ H_1$  and  $\Omega_n = G_n^{-1}(D_n)$ . Then the sequence  $G_n$  converges, uniformly on compact sets on  $\Omega = \cup_{n=1}^{\infty} \Omega_n$  to a map  $G$  which maps  $\Omega$  biholomorphically onto  $\mathbb{C}^2$ .*

**Proof of Lemma 3.2.2.** [Globevnik, 1998]

Observe that for all  $n \in \mathbb{N}$ ,  $\Omega_n \subset\subset \Omega_{n+1}$ :

$$H_{n+1}(D_n) \subset\subset D_{n+1} \Rightarrow D_n \subset\subset H_{n+1}^{-1}(D_{n+1}) \Rightarrow G_n^{-1}(D_n) \subset\subset G_{n+1}^{-1}(D_{n+1})$$

i.e.  $\Omega_n \subset\subset \Omega_{n+1}$ . Let  $n \in \mathbb{N}$  and  $z \in \Omega_n$  and  $k > m \geq n$ . Then writing  $w = G_m(z) \in D_m$ , a) and b)



imply

$$\begin{aligned}
|G_k(z) - G_m(z)| &= |H_k \circ H_{k-1} \circ \cdots \circ H_{m+1}(w) - w| \\
&\leq |w - H_{m+1}(w)| + |H_{m+1}(w) - H_{m+2}(H_{m+1}(w))| \\
&\quad + \cdots + |H_{k-1} \circ \cdots \circ H_{m+1}(w) - H_k(H_{k-1} \circ \cdots \circ H_{m+1}(w))| \\
&< \frac{\varepsilon_m}{2^m} + \frac{\varepsilon_{m+1}}{2^{m+1}} + \cdots + \frac{\varepsilon_{k+1}}{2^{k+1}} \\
&< \frac{\varepsilon_n}{2^{n-1}}.
\end{aligned}$$

Therefore  $G_m$  converges uniformly on  $\Omega_n$  to a holomorphic map  $G$ , where  $|G(z) - G_n(z)| < \varepsilon_n$ , for all  $z \in \Omega_n$ . That is,  $|G \circ G_n^{-1}(w) - w| < \varepsilon_n$  for all  $w \in D_n$ , so by c),  $G \circ G_n^{-1}$  is injective on  $H_n(D_{n-1})$ , i.e.  $G$  is injective on  $G_n^{-1} \circ H_n(D_{n-1}) = H_1^{-1} \circ \cdots \circ H_n^{-1} \circ H_n(D_{n-1}) = G_{n-1}^{-1}(D_{n-1}) = \Omega_{n-1}$ . Therefore  $G$  maps  $\Omega$  biholomorphically onto  $G(\Omega)$ .

We now prove  $G(\Omega) = \mathbb{C}^2$ . Denote by  $\mathbb{B}$  the open unit ball of  $\mathbb{C}^2$ . Since  $D_n$ ,  $n \in \mathbb{N}$  form an increasing union up to  $\mathbb{C}^2$  and the sequence  $\varepsilon_n$  is decreasing, there exist a sequence  $r_n \nearrow \infty$  and an integer  $n_0$  such that

$$r_n > 2\varepsilon_n \quad \text{and} \quad r_n \mathbb{B} \subset D_n$$

for each  $n \geq n_0$ . We will demonstrate that  $(r_n - \varepsilon_n)\mathbb{B} \subset G(\overline{\Omega_n})$ , for  $n \geq n_0$ , hence  $G(\Omega)$  contains arbitrarily large balls centered at the origin, i.e.  $G(\Omega)$  is  $\mathbb{C}^2$ . Let  $F_n = G \circ G_n^{-1}$ . We will prove  $(r_n - \varepsilon_n)\mathbb{B} \subset F_n(\overline{D_n})$  for all  $n \geq n_0$ . The set  $L_n = F_n(\overline{D_n}) \cap (r_n - \varepsilon_n)\mathbb{B}$  is closed in  $(r_n - \varepsilon_n)\mathbb{B}$ . We already showed that  $|F_n(w) - w| \leq \varepsilon_n$  for all  $w \in D_n$ , so in particular,  $|F_n(0)| \leq \varepsilon_n$ . Since  $0 \in D_n$  and  $\varepsilon_n < r_n - \varepsilon_n$ , we conclude  $F_n(0) \in L_n$  and hence  $L_n \neq \emptyset$ . Let  $w \in L_n$ . Then  $w \in (r_n - \varepsilon_n)\mathbb{B}$  and  $w = G_n(z)$ , where  $z \in \overline{D_n}$  and  $|w - z| \leq \varepsilon_n$ , so  $z \in r_n \mathbb{B} \subset D_n$ . Because  $D_n$  are domains there exists a neighborhood  $U(z)$  of  $z$  such that  $U(z) \subset D_n$ . Since  $F_n$  is an open map we conclude  $F_n(U(z)) \cap (r_n - \varepsilon_n)\mathbb{B}$  is a neighborhood of  $w$  contained in  $L_n$ . Therefore  $L_n$  is open and since  $(r_n - \varepsilon_n)\mathbb{B}$  is connected,  $L_n = (r_n - \varepsilon_n)\mathbb{B}$  and hence  $(r_n - \varepsilon_n)\mathbb{B} \subset F_n(\overline{D_n})$ .  $\square$

When we apply lemma 3.2.2, to show part c) is satisfied we will use the following lemma due

to Narasimhan. Recall that a holomorphic map is called regular at a point  $x$  in its domain if the Jacobian matrix of  $f$  at  $x$  has full rank. If  $f$  is regular on its domain we say  $f$  is regular.

**Lemma 3.2.3.** *[Narasimhan, 1960] If  $X$  is a domain in  $\mathbb{C}^2$  and  $f : X \rightarrow \mathbb{C}^2$  is holomorphic, injective, and regular in a neighborhood of a compact set  $K \subset X$  and if  $g : X \rightarrow \mathbb{C}^2$  is holomorphic and satisfies  $|f - g| < \varepsilon$  on a neighborhood  $K'$  of  $K$ , then if  $\varepsilon$  is small enough,  $g$  is injective and regular on  $K$ .*

**Proof of Lemma 3.2.3.** [Narasimhan, 1960]

Let  $D$  denote the diagonal of  $\mathbb{C}^2$ . There is some  $\varepsilon_0 > 0$  such that if  $\varepsilon < \varepsilon_0$ , then  $g$  is regular on  $K$ . Moreover, if  $\varepsilon_0$  is small enough, then there is a neighborhood of any point of  $K$ , depending only on  $\varepsilon_0$ , such that  $g$  is injective on this neighborhood. Therefore there exists a neighborhood  $U$  of  $(K \times K) \cap D$ , depending only on  $\varepsilon_0$ , so that if  $|f - g| < \varepsilon_0$  on  $K'$ , and  $(x, y) \in U \setminus D$ , then  $g(x) \neq g(y)$ . Let  $\delta = \inf_{(x,y) \in K \times K \setminus U} |f(x) - f(y)|$ . Then  $\delta > 0$ . If  $\varepsilon_0$  is small enough then  $|g(x) - g(y)| \geq \frac{\delta}{2} > 0$  if  $(x, y) \in (K \times K) \setminus U$ .  $\square$

Theorem 6 has two main parts, namely  $\Omega$  is Fatou-Bieberbach and  $\Omega$  has a certain boundary smoothness. The push out method allows us to construct Fatou-Bieberbach domains (in particular the domain of theorem 6) with enough freedom to control their boundary smoothness, though we will need more tools to make use of this freedom to control the boundary.

### 3.3 Existence of Fatou-Bieberbach Domains with Locally $C^\ell$ -Smooth Boundary

In this section we will construct a Fatou-Bieberbach domain  $\Omega$  in  $\mathbb{C}^2$  with certain boundary smoothness properties. We will design  $\Omega$  so that it has  $C^k$ -smooth boundary in a neighborhood of the origin. In fact, we may choose any radius  $R > 0$  and construct our domain so that in the polydisc about the origin, of polyradius  $(R, R)$ , the domain  $\Omega$  is an arbitrarily small perturbation of the polydisc centered at the origin of polyradius  $(1, R)$ . The part of  $\mathbb{C}^2$  not contained in  $\Omega$  will be

unbounded. Let  $\Delta = \{z \in \mathbb{C} : |z| < 1\}$ , and for  $R > 0$ ,  $R\Delta = \{z \in \mathbb{C} : |z| < R\}$ . Our main theorem this chapter is the following:

**Theorem 5 (C).** *Let  $1 < R < \infty$  and  $\ell \in \mathbb{N}, \ell \geq 1$ . There exist a domain  $\Omega \subset \mathbb{C}^2$  such that*

1.  $\Omega \subset \{(z, w) : |z| < \max\{R, |w|\}\}$
2.  $\Omega \cap (R\Delta \times R\Delta)$  is an arbitrarily small  $C^\ell$ -perturbation of  $\Delta \times R\Delta$
3. There exists a volume-preserving biholomorphic map from  $\Omega$  onto  $\mathbb{C}^2$ .

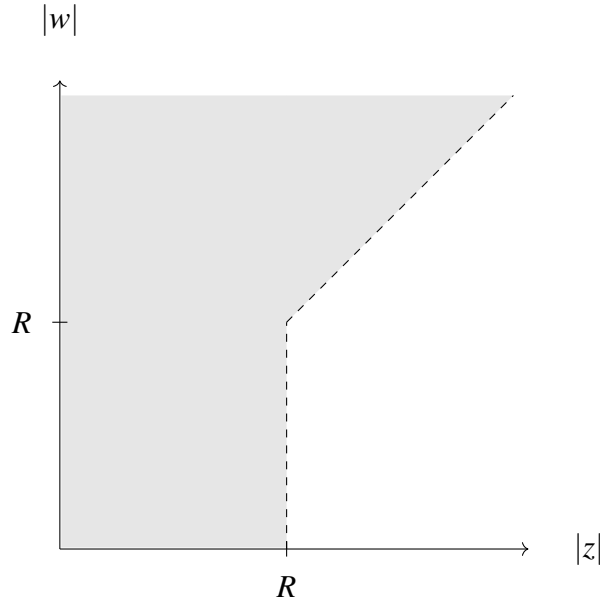


Figure 3.1: **The set**  $\{(z, w) : |z| < \max\{R, |w|\}\}$

$\Omega$  is contained in the shaded area in figure 3.1 above. Globevnik proved this theorem in the case  $k = 1$  [Globevnik, 1998]. Our proof has two main parts, and is similar to Globevnik's but we address fundamental changes required to increase smoothness beyond the  $C^1$  case. The first part is showing that  $\Omega$  is a Fatou-Bieberbach domain satisfying the condition  $\Omega \subset \{(z, w) : |z| < \max\{R, |w|\}\}$ . The second part is showing that the boundary of  $\Omega$  in the set  $R\Delta \times R\Delta$  has the claimed smoothness.

To prove theorem 5, we begin with three "Push-out" lemmas: lemma 3.2.2, lemma 3.2.3, lemma 3.4.1. These lemmas allow us to ensure that  $\Omega$  is a Fatou-Bieberbach domain contained

in the set  $\{(z, w) : |z| < \max\{R, |w|\}\}$ . Afterwards we establish two boundary smoothness lemmas: lemma 3.5.1, lemma 3.5.2. These lemmas allow us to ensure that  $\Omega$  has locally  $C^\ell$ -smooth boundary, and has the almost cylinder shape. Finally we use a limit process to complete the proof.

### Outline of the Proof theorem 5

Let  $\Omega$  denote the FB-domain to be constructed, with a biholomorphism  $\Phi : \Omega \rightarrow \mathbb{C}^2$ . The theorem will show that

$$\Omega \cap (R\Delta \times R\Delta) \text{ is an arbitrarily small } C^k\text{-perturbation of } \Delta \times R\Delta \quad (3.3.1)$$

and

$$\Omega \subset \{(z, w) : |z| < \max\{R, |w|\}\}. \quad (3.3.2)$$

Notice that (3.3.2) can also be characterized by:

$$E = \cup_{|a| \leq 1} \{(\xi, a\xi) : |\xi| \geq R\} \text{ and } E \cap \Omega = \emptyset. \quad (3.3.3)$$

For the limit process we will choose

$I_1$ ) Sequences  $\{T_n\}, \{V_n\}$  of positive integers increasing to  $\infty$ .

$I_2$ ) Sequences  $\{\epsilon_n\}, \{\tau_n\}$  of decreasing positive numbers.

$I_3$ ) A Sequence  $\{M_n\}$  of positive integers.

If we choose these sequences appropriately (in particular so that lemma 3.4.1 and lemma 3.2.2 are satisfied) and we define:

$$\left\{ \begin{array}{ll} S_n(z, w) = \left( z, w + V_n \left( \frac{z}{T_{n-1}} \right)^{M_n} \right) & \text{if } n \text{ is odd} \\ S_n(z, w) = \left( z + T_n \left( \frac{w}{V_{n-1}} \right)^{M_n}, w \right) & \text{if } n \text{ is even} \end{array} \right\} \quad (3.3.4)$$

and

$$D_n = \{(z, w) : |z| < T_n - \tau_n, |w| < V_n - \tau_n\} \quad (3.3.5)$$

Then the push-out hypotheses from lemma 3.2.2 are met and

$$D_n \cap (S_n \circ S_{n-1} \circ \cdots \circ S_1)(E) = \emptyset \quad (3.3.6)$$

for every  $n \in \mathbb{N}$ .

Once this is done, the push-out method shows that  $\Phi = \lim_{n \rightarrow \infty} S_n \circ \cdots \circ S_1$  maps the domain  $\Omega = \cup \Omega_n$  biholomorphically onto  $\mathbb{C}^2$ . It can be seen from (3.3.6) that  $\Omega$  is not all of  $\mathbb{C}^2$ , and in fact (3.3.2) holds.

During the limit process we will also take care to control the sequences  $\{T_n\}, \{V_n\}, \{M_n\}$  so that the boundary smoothness lemmas ensure the boundaries of the approximating sets in the polydisc  $R\Delta \times R\Delta$  are  $C^\ell$  smooth and converge in  $R\Delta \times R\Delta$  to a  $C^\ell$  smooth boundary for  $\Omega$  in  $R\Delta \times R\Delta$ .

### 3.4 Push-Out Lemmas

To ensure the domain  $\Omega$  of theorem 5 is in fact a Fatou-Bieberbach domain and  $\Omega \subset \{(z, w) : |z| < \max\{R, |w|\}\}$  we customize a version of push-out method due to Globevnik [Globevnik, 1998], using stronger boundary smoothness lemmas. The method will use two lemmas. Lemma 3.2.2 ensures that  $\Omega$  is a Fatou-Bieberbach Domain, while lemma 3.4.1 ensures that enough of  $\mathbb{C}^2$  is "pushed out" of  $\Omega$ , i.e.  $\Omega \subset \{(z, w) : |z| < \max\{R, |w|\}\}$ . Before we state lemma 3.4.1, we need a few definitions.

Let  $\Phi_n = S_n \circ \cdots \circ S_1$  and let  $(P(\xi, a), Q(\xi, a)) = \Phi_n(\xi, a\xi)$ . To ensure (3.3.2) we will require

that

$$\begin{cases} |P(\xi, a)| \geq 2T_n & \text{for } (|\xi| \geq R, |a| \leq 1) & \text{if } n \text{ is odd} \\ |Q(\xi, a)| \geq 2V_n & \text{for } (|\xi| \geq R, |a| \leq 1) & \text{if } n \text{ is even.} \end{cases} \quad (3.4.1)$$

Let

$$\Phi_n(\xi, a\xi) = (P_n(\xi, a), Q_n(\xi, a)).$$

Let  $\mathcal{A}$  denote the set of all functions of the form

$$p(\xi, a) = p_0\xi^k + p_1(a)\xi^{k-1} + \cdots + p_k(a)$$

where  $k \geq 1$ ,  $p_0$  is a nonzero constant and  $p_1, \cdots, p_k$  are polynomials.

Notice that If  $p, q \in \mathcal{A}$  then for all large enough integers  $M$ ,

$$p + T \left( \frac{q}{r} \right)^M \in \mathcal{A}, \quad q + T \left( \frac{q}{r} \right)^M \in \mathcal{A}$$

for each  $T > 0, r > 0$ .

**Lemma 3.4.1** (Polynomial Push Out). *Let  $P, Q \in \mathcal{A}$ , let  $S > 0$ ,  $R > 0$  and assume that for  $|\xi| \geq R, |a| \leq 1$ ,  $|P(\xi, a)| \geq S$ . Given  $T > 0$  and  $s$ ,  $0 < s < S$ ,*

*there is an integer  $N_0$  such that for  $N \geq N_0$ ,*

$$\left| \frac{Q(\xi, a)}{T} + \left( \frac{P(\xi, a)}{s} \right)^N \right| \geq 2. \quad (3.4.2)$$

For  $|\xi| \geq R, |a| \leq 1$ .

**Proof of Lemma 3.4.2.**

We will prove that  $\left| \left( \frac{P(\xi, a)}{s} \right)^N \right| \geq 2 + \left| \frac{Q(\xi, a)}{T} \right|$ . First let

$$P(\xi, a) = P_0\xi^m + p_1(a)\xi^{m-1} + \cdots + p_m(a)$$

and

$$Q(\xi, a) = Q_0 \xi^n + Q_1(a) \xi^{n-1} + \cdots + Q_n(a).$$

Choose  $k \in \mathbb{N}$  large so that  $\ell = mk \geq n$ . Let

$$\left( \frac{P(\xi, a)}{s} \right)^k = \left( \frac{P_0}{s} \right)^k \xi^\ell + P_1(a) \xi^{\ell-1} + \cdots + P_\ell(a)$$

Let  $A = \max_{|a| \leq 1, j=\{1,2,\dots,\ell\}} \{|P_j(a)|\}$ ,  $B = \max_{|a| \leq 1, j=\{1,2,\dots,n\}} \{|Q_j(a)|\}$ .

$$\begin{aligned} \left| \frac{P(\xi, a)}{s} \right|^k &\geq \left( \frac{|P_0|}{s} \right)^k |\xi|^\ell - A(|\xi|^{\ell-1} + \cdots + 1) \\ |Q(\xi, a)| &\leq |Q_0| |\xi|^n + B(|\xi|^{n-1} + \cdots + 1). \end{aligned}$$

Choose  $\rho_0 \geq R$  so that on  $\{|\xi| \geq \rho_0, |a| \leq 1\}$ ,  $|Q(\xi, a)| \geq 1$ . Then

$$\begin{aligned} \frac{\left| \frac{P(\xi, a)}{s} \right|^k}{|Q(\xi, a)|} &\geq \frac{\left( \frac{|P_0|}{s} \right)^k |\xi|^\ell - A(|\xi|^{\ell-1} + \cdots + 1)}{|Q_0| |\xi|^n + B(|\xi|^{n-1} + \cdots + 1)} \\ &= |\xi|^{\ell-n} \frac{\left( \frac{|P_0|}{s} \right)^k - A(|\xi|^{-1} + \cdots + |\xi|^{-\ell})}{|Q_0| + B(|\xi|^{-1} + \cdots + |\xi|^{-n})} \end{aligned}$$

and for some  $\rho_1 \geq \rho_0$  this last quantity is larger than  $2 + \frac{1}{T}$  on the set  $\{|\xi| \geq \rho_1, |a| \leq 1\}$ .

For the set  $\{R \leq |\xi| \leq \rho_1, |a| \leq 1\}$ , let  $M = \max_{\{R \leq |\xi| \leq \rho_1, |a| \leq 1\}} \{|Q(\xi, a)|\}$ . Then choose  $N_0$  large so that if  $N \geq N_0 \geq k$

$$\left| \left( \frac{P(\xi, a)}{s} \right)^N \right| \geq 2 + \frac{M}{T}.$$

which easily follows from the fact that  $\left| \frac{P(\xi, a)}{s} \right| > 1$  on the set  $\{R \leq |\xi| \leq \rho_1, |a| \leq 1\}$ . Lastly note that increasing  $T$  preserves the inequality

$$\left| \left( \frac{P(\xi, a)}{s} \right)^N \right| \geq 2 + \left| \frac{Q(\xi, a)}{T} \right|, \text{ and therefore by the proof preserves (3.4.2).}$$

□

We note that Lemma 3.4.1 and its proof are modified versions of lemma 4.1 and its proof in

[Globevnik, 1998].

### 3.5 Showing Boundary Smoothness

Our argument has two main steps. Lemma 3.5.1 allows us to control boundary smoothness of the  $\Omega_{n+1}$  domain in the  $(z_n, w_n)$  coordinates, while lemma 3.5.2 allows us to preserve boundary smoothness when changing to other coordinates  $(z_k, w_k)$ . We begin with some definitions for the precise smoothness conditions under consideration. For each  $\ell \in \mathbb{N}$  let  $C^\ell(\partial\Delta \times R\bar{\Delta})$  be the Banach space  $C^\ell(\partial\Delta \times R\bar{\Delta})$  of real functions  $f(e^{i\theta}, x)$  on  $\mathbb{R} \times R\bar{\Delta}$  such that

1.  $f$  has derivatives of order up to  $\ell$  on the interior of  $\partial\Delta \times R\bar{\Delta}$ .
2.  $f$  has continuous extension to  $\mathbb{R} \times R\bar{\Delta}$ .
3.  $f$  has norm

$$\|f\|_{C^\ell(\partial\Delta \times R\bar{\Delta})} = \sum_{|\beta| \leq \ell} \sup_{\mathbb{R} \times R\bar{\Delta}} \left| D^\beta f(e^{i\theta}, x_1, x_2) \right|.$$

**Lemma 3.5.1.** *Let  $0 < R < \infty$ . Given  $M \in \mathbb{N}, \ell \in \mathbb{N}$  there exists  $\alpha_0 > 0$  such that for every real  $\alpha$ ,  $0 < \alpha < \alpha_0$ , there is a function  $\phi_\alpha \in C^\ell(\partial\Delta \times R\bar{\Delta})$  such that*

$$\begin{aligned} D &= \{(z, w) \in \mathbb{C}^2 : |z^M + \alpha w| = 1, |w| \leq R\} \\ &= \{(\phi_\alpha(\xi, w)\xi, w) : \xi \in \partial\Delta, |w| \leq R\} \end{aligned}$$

*and there exists a constant  $C$  depending on  $\ell$  such that*

$$\|\phi_\alpha - 1\|_{C^\ell(\partial\Delta \times R\bar{\Delta})} < \alpha C M^{2\ell}.$$

The proof of lemma 3.5.1 will be presented after the proof of Theorem 5.

Lemma 3.5.1 says that  $D$  "looks" like a cylinder, and allows us to control its smoothness by the choices of the constants  $M$  and  $\alpha$ . The second lemma will allow us to preserve this boundary



smoothness through the limit process. In order to state the second lemma we first need definitions to clarify the nature of the sets whose smoothness we are tracking. We note that while Lemma 3.5.1 looks similar to Lemma 5.1 in [Globevnik, 1998], in Lemma 3.5.1 we gain  $C^\ell$ -smoothness at the cost of being dependent on the exponent  $M$  of the shear. Let  $n \in \mathbb{N}, 0 < R \in \mathbb{R}$  and define

$$P^n(R) = \{|z_n| = 1, |w_n| \leq R\}.$$

A set  $G \subset \mathbb{C}^2$  will be called the  $C^k$ -graph over  $P^n(R)$ , given by the function  $r$ , if  $r : P^n(R) \rightarrow \mathbb{R}_+$  is in the Banach space  $C^k(P^n(R))$  and

$$G = \{(r(\xi, w_n)\xi, w_n) : |\xi| = 1, |w_n| \leq R\}.$$

For  $R > 0$ , let  $D_R = \{(z, w) \in \mathbb{C}^2 : |z| \leq R, |w| \leq R\}$ .

**Lemma 3.5.2.** [Globevnik, 1997] *Let  $r > 0$  and  $k \in \mathbb{N}$ . Let  $F$  be a holomorphic automorphism of  $\mathbb{C}^2$  and let  $R > 0$  be so large that  $D_r \subset \subset F(D_R)$ . Let  $0 < \alpha < R$ , and  $S = \{(z, w) \in D_R : |z| = \alpha\}$  and assume that  $F(S) \cap D_r$  is a  $C^k$ -graph over  $P(r)$ . Given  $\varepsilon > 0$ , there is a  $\delta > 0$  such that if  $T$  is a graph over  $P(r)$  in the  $\delta$ -neighborhood of  $S$  then  $F(T) \cap D_r$  is a  $C^k$ -graph over  $P(r)$  belonging to the  $\varepsilon$ -neighborhood of  $F(S) \cap D_r$ .*

**Proof of Lemma 3.5.2.** [Globevnik, 1997]

By the assumptions, for all points of the form  $(s, w)$  with  $|s| = 1, |w| = r$ ,  $F(S \cap \text{Int} D_R)$ , is a closed submanifold of  $F(\text{Int} D_R)$ , and is transverse to the ray  $\{(ts, w) : t > 0\}$ . By compactness there exists  $r' > r$ , with  $D_{r'} \subset \subset F(D_R)$ , such that  $F(S) \cap D_{r'}$  is still a  $C^k$ -graph over  $P(r)$ . Thus a small enough  $C^k$ -perturbation of  $F(S)$  will intersect  $D_r$  in a set of the form  $L \cap D_r$  where  $L$  is a small  $C^k$ -perturbation of  $F(S) \cap D_{r'}$ . Provided the perturbation is sufficiently small,  $L \cap D_r$  is a  $C^k$ -graph over  $P(r)$  arbitrarily close to  $F(S) \cap D_r$ .  $\square$

We not that strictly speaking, Globevnik stated and proved the  $k = 1$  case of Lemma 3.5.2 but the argument here is essentially the same.

**Proof of Theorem 5.**

Our limit process will be guided by the needs to satisfy the push-out hypothesis at each stage so that  $\Omega$  is a Fatou Bieberbach domain and the need to maintain enough boundary smoothness at each stage so that  $\Omega$  restricted to the polydisc  $R\Delta \times R\Delta$  has a smooth boundary in the limit. Recall that the argument will require

$I_1)$  Sequences  $\{T_n\}, \{V_n\}$  of positive integers increasing to  $\infty$ .

$I_2)$  Sequences  $\{\varepsilon_n\}, \{\tau_n\}$  of decreasing positive numbers.

$I_3)$  A Sequence  $\{M_n\}$  of positive integers.

$I_4)$   $\phi_n \in C^k(\partial\Delta \times R\bar{\Delta})$ .

such that, for all  $n \in \mathbb{N}$ , The push out conditions:

(a)  $\text{dist}(D_{n-1}, \mathbb{C}^2 \setminus D_n) > \frac{\varepsilon_n}{2^n}$

(b)  $|S_{n+1}(z, w) - (z, w)| < \frac{\varepsilon_n}{2^n}$  For  $z \in D_n$

(c) If  $\Phi : D_n \rightarrow \mathbb{C}^2$  satisfies  $|\Phi(z, w) - (z, w)| < \varepsilon_n$  For  $z \in D_n$ , then  $\Phi$  is bijective on  $S_n(D_{n-1})$

and the boundary smoothness conditions:

1. 
$$\begin{cases} |P(\xi, a)| \geq 2T_n & \text{for } (|\xi| \geq R, |a| \leq 1) & \text{if } n \text{ is odd} \\ |Q(\xi, a)| \geq 2V_n & \text{for } (|\xi| \geq R, |a| \leq 1) & \text{if } n \text{ is even.} \end{cases}$$
2.  $\Omega_n \cap (R\Delta \times R\Delta) = \{(t\xi, w) : 0 \leq t < \phi_n(\xi, w), \xi \in \partial\Delta, |w| < R\}$
3.  $\|\phi_n - \phi_{n-1}\|_{C^k(\partial\Delta \times R\Delta)} < \frac{\eta}{2^n}$ , for some fixed  $0 < \eta < \frac{1}{2}$ .

are satisfied. Choose  $0 < \eta < \frac{1}{2}$ . Let  $D_0 = \emptyset$ ,  $T_0 = 1 - \eta$ ,  $\phi_0 = 1 - \eta$ . Notice that  $M \geq 2$ ,  $V_1 \geq 1$  implies

$$P_1 = a\xi + V_1 \left( \frac{\xi}{T_0} \right)^M \in \mathcal{A} \quad \text{and} \quad Q_1 = a\xi \in \mathcal{A}.$$

There exists  $\tilde{M} \geq 2$  and  $\tilde{V} \geq 1$  such that if  $V_1 \geq \tilde{V}$ ,  $M \geq \tilde{M}$ , then

$$\left| a\xi + V_1 \left( \frac{\xi}{T_0} \right)^M \right| \geq 2V_1 \quad \text{for } |\xi| \geq R, |a| \leq 1. \quad (3.5.1)$$

Explicitly,

$$\left| a\xi + V_1 \left( \frac{\xi}{T_0} \right)^M \right| \geq \frac{1}{T_0^M} |\xi|^M - |\xi| = |\xi| \left( \frac{1}{T_0^M} |\xi|^{M-1} - 1 \right)$$

and

$$\left( \frac{1}{T_0^M} |\xi|^{M-1} - 1 \right) \geq (|\xi|^{M-1} - 1) \geq R^{M-1} - 1.$$

Since  $R > 1$ , one can make this last expression as large as desired by taking  $M$  large enough. Let  $M_1$  be some such large enough  $M$ . Since it is assumed that  $|\xi| \geq R > 1$ , the inequality (3.5.1) follows.

Now to ensure that  $\|\phi_1 - \phi_0\|_{C^k(\partial\Delta \times R\Delta)} < \frac{\eta}{2} = \frac{\eta}{2}$ , The smoothness lemma 3.5.1 requires that

$$V_1 > \frac{2}{\eta} C_1(k) \left( \frac{M_1!}{(M_1 - k)!} \right)^2 \quad (3.5.2)$$

while the push-out condition b)

$$|S_1(z, w) - (z, w)| < \frac{\varepsilon_1}{2} \quad \text{For } z \in D_0$$

is trivially satisfied because  $D_0 = \emptyset$ . So choose  $V_1$  large enough to satisfy (3.5.2). Then there is a function  $\tilde{\phi} \in C^k(\partial\Delta \times R\bar{\Delta})$  such that:

$$\begin{aligned} \{(z, w) : \left| w + V_1 \left( \frac{z}{T_0} \right)^{M_1} \right| < V_1, |w| < R\} \\ = \{(t\xi, w) : 0 \leq t < \tilde{\phi}(\xi, w), |\xi| = 1, |w| < R\} \end{aligned}$$

and  $\left\| \tilde{\phi} - \phi_0 \right\|_{C^k(\partial\Delta \times R\Delta)} < \frac{\eta}{2}$  and then choose  $\tau_1 > 0$  small so that:

$$\begin{aligned} \{(z, w) : \left| w + V_1 \left( \frac{z}{T_0} \right)^{M_1} \right| < V_1 - \tau_1, |w| < R\} \\ = \{(t\xi, w) : 0 \leq t < \phi_1(\xi, w), |\xi| = 1, |w| < R\} \end{aligned}$$

where  $\phi_1 \in C^k(\partial\Delta \times R\bar{\Delta})$  and

$$\left\| \phi_1 - \tilde{\phi} \right\|_{C^k(\partial\Delta \times R\Delta)} < \frac{\eta}{16}.$$

Now choose  $T_1$  large enough that  $T_1 - \tau_1 > R$ . It follows that since

$$S_1 = \left( z, w + V_1 \left( \frac{z}{T_0} \right)^{M_1} \right),$$

$$\Omega_1 = S_1^{-1}(D_1) = \left\{ (z, w) : |z| < T_1 - \tau_1, \left| w + V_1 \left( \frac{z}{T_0} \right)^{M_1} \right| < V_1 - \tau_1 \right\} \text{ and}$$

$\Omega_1 \cap (R\Delta \times R\Delta) = \{(t\xi, w) : 0 \leq t < \phi_1(\xi, w), |\xi| = 1, |w| < R\}$ . Set  $\varepsilon_1 = \varepsilon_0 = 1$ . The push-out hypotheses c) for  $n = 1$  and a) and b) for  $n = 0$  are then satisfied.

Assume now that  $n$  is odd and  $T_j, V_j, \varepsilon_j, \tau_j, M_j, \phi_j, 1 \leq j \leq n$  have been chosen so that if  $P_n, Q_n \in \mathcal{A}$ , (3.4.1), the condition  $\Omega_n \cap (R\Delta \times R\Delta) = \{(t\xi, w) : 0 \leq t < \phi_n(\xi, w), \xi \in \partial\Delta, |w| < R\}$ , b) from the push-out lemma and the inequalities  $\left\| \phi_j - \phi_{j-1} \right\|_{C^k(\partial\Delta \times R\Delta)} < \frac{\eta}{2^j}$  are satisfied, and for  $\Phi_n = (F_n, G_n)$ ,

$$\begin{aligned} \{(z, w) : |G_n| < V_n\} \cap (R\Delta \times R\Delta) \\ = \{(t\xi, w) : 0 \leq t < \psi_n(\xi, w), |\xi| = 1, |w| < R\} \end{aligned}$$

where  $\psi_n \in C^1(\partial\Delta \times R\bar{\Delta})$  and

$$\left\| \phi_n - \psi_n \right\|_{C^k(\partial\Delta \times R\Delta)} < \frac{\eta}{2^{n+3}}. \quad (3.5.3)$$

Now choose  $\tilde{M}_{n+1} > M_n$  so that if  $T_{n+1} > 2T_n$ ,

$$\left| \frac{P_n(\xi, a)}{T_{n+1}} + \left( \frac{Q_n(\xi, a)}{V_n} \right)^{\tilde{M}_{n+1}} \right| \geq 2$$

on the set  $\{|a| \leq 1, |\xi| \geq R\}$  and  $P_{n+1}, Q_{n+1} \in \mathcal{A}$ . Denote by  $\mathbb{B}$  the open unit ball in  $\mathbb{C}^2$ . Choose  $t > 2R$  large enough that the surface  $\{(z, w) : |G_n(z, w)| = V_n\}$  intersects the sphere  $t\partial\mathbb{B}$  transversely. If  $T_{n+1} > 2T_n$  is large enough then the boundary lemma (lemma 3.5.1) in  $Z, W$  coordinates with  $Z = F_n(z, w), W = G_n(z, w)$  gives a range of  $M$  values for which the surface

$$\left\{ \left| F_n(z, w) + T_{n+1} \left( \frac{G_n(z, w)}{V_n} \right)^M \right| = T_{n+1} \right\} \cap t\mathbb{B}$$

is a small  $C^k$ -perturbation of

$$\{(z, w) : |G_n(z, w)| = V_n\} \cap t\mathbb{B}.$$

Specifically, by lemma 3.5.2, If  $T_{n+1} > 2T_n$  is large enough (and  $M$  is large enough) then there is a function  $\phi_{n+1, M} \in C^k(\partial\Delta \times R\bar{\Delta})$  for which

$$\|\phi_{n+1, M} - \psi_n\|_{C^k(\partial\Delta \times R\bar{\Delta})} < \frac{\eta}{2^{n+3}} \quad (3.5.4)$$

and

$$\begin{aligned} & \left\{ (z, w) : \left| F_n(z, w) + T_{n+1} \left( \frac{G_n(z, w)}{V_n} \right)^m \right| < T_{n+1} \right\} \cap (R\Delta \times R\bar{\Delta}) \\ &= \{(t\xi, w) : 0 \leq t < \phi_{n+1, M}(\xi, w), |\xi| = 1, |w| < R\}. \end{aligned}$$

On the other hand there is an  $(M, T)$  range (a nonempty subset of  $\mathbb{N}^2$ ) for which the condition

$$|S_{n+1}(z, w) - (z, w)| < \frac{\varepsilon_n}{2^n} \quad \text{For } z \in D_n$$

is satisfied. In other words, For boundary smoothness the smoothness lemma (lemma 3.5.1) re-

quires:

$$T_{n+1} > \frac{2^{n+3}}{\eta} C_{n+1}(k) \left( \frac{M_{n+1}!}{(M_{n+1}-k)!} \right)^2$$

we will use the slightly stronger inequality

$$T_{n+1} > \frac{2^{n+3}}{\eta} C_{n+1}(k) (M_{n+1})^{2k} \quad (3.5.5)$$

while to obtain b) of the push-out lemma it is sufficient that:

$$T_{n+1} < \varepsilon_n \left( \frac{V_n}{V_n - \tau_n} \right)^{M_{n+1}} \quad (3.5.6)$$

Since, for  $t > 1$ ,  $\lim_{M \rightarrow \infty} \frac{M^{2k}}{t^M} = 0$ , one can choose  $M_{n+1} > M_n$  and  $T_{n+1} > 2T_n$  so large that both inequalities (3.5.5) and (3.5.6) are satisfied.

Now choose  $\tau_{n+1}, 0 < \tau_{n+1} < \tau_n$  small enough that

$$S_{n+1}(\overline{D_n}) \subset \{(z, w) : |z| < T_{n+1} - \tau_{n+1}\}$$

and  $F_{n+1}(z, w) = F_n(z, w) + T_{n+1} \left( \frac{G_n(z, w)}{V_n} \right)^{M_{n+1}}$  satisfies

$$\begin{aligned} \{(z, w) : |F_{n+1}| < T_{n+1} - \tau_{n+1}\} \cap (R\Delta \times R\Delta) \\ = \{(t\xi, w) : 0 \leq t < \phi_{n+1}(\xi, w), |\xi| = 1, |w| < R\} \end{aligned}$$

where

$$\|\phi_{n+1} - \psi_{n+1}\|_{C^k(\partial\Delta \times R\Delta)} < \frac{\eta}{2^{n+4}}.$$

Now choose  $V_{n+1} > 2V_n$  large so that

$$\{(z, w) : |F_{n+1}| < T_{n+1} - \tau_{n+1}\} \cap (R\Delta \times R\Delta)$$

is contained in the set

$$\{(z, w) : |G_n(z, w)| < V_{n+1} - \tau_{n+1}\}$$

and  $S_{n+1}(\overline{D_n}) \subset \{(z, w) : |w| < V_{n+1} - \tau_{n+1}\}$ . Then it follows that

$$\begin{aligned} \Omega_{n+1} \cap (R\Delta \times R\Delta) \\ = \{(t\xi, w) : 0 \leq t < \phi_{n+1}(\xi, w), |\xi| = 1, |w| < R\}. \end{aligned}$$

Now using lemma 3.2.3 with  $f = \text{Id}_{\mathbb{C}^2}$ , the identity map on  $\mathbb{C}^2$ ,  $K = \overline{D_{n+1}}$  and  $K' = H_{n+1}(\overline{K_n})$  choose  $\varepsilon_{n+1}, 0 < \varepsilon_{n+1} < \varepsilon_n$  so small that (c) in the push-out lemma is satisfied with  $(n+1)$  in the place of  $n$ .

The proof for the case  $n$  is even is the same if the coordinates are swapped. Lastly, the shears  $\{S_n\}$  have Jacobian determinant one, and hence so does any finite composition of the  $S_n$ 's. For example, if  $n$  is odd,  $S_n(z, w) = \left(z + T_n\left(\frac{w}{V_{n-1}}\right)^{M_n}, w\right)$ , so the Jacobian determinant is

$$\begin{vmatrix} 1 & \frac{T_n M_n}{V_{n-1}} \left(\frac{w}{V_{n-1}}\right)^{M_n-1} \\ 0 & 1 \end{vmatrix} = 1.$$

Therefore, by the continuity of the determinant, the biholomorphism  $F$  is volume preserving. Thus theorem 5 is proven.  $\square$

*Remark 3.* The domains  $\Omega_j$  in the proof of theorem 5 live in  $\mathbb{C}^2$ , which we may view as  $\mathbb{R}^4$ . In order to try to understand these domains better, we now present some pictures of real three dimensional slices of the domain  $\widetilde{\Omega}_1 = \{(z, w) \in \mathbb{C}^2 : |z^2 + .1w| = 1\}$ . In each figure, we have set the  $w$  variable equal to 1 and we are looking at the part the real three dimensional set:  $\widehat{\Omega}_1 = \{z \in \mathbb{C}, t \in \mathbb{R} : |z^2 + .1t| = 1\}$  restricted to a bounded set centered about the origin.

Notice that in the first figure, the slice looks like a real circular cylinder. As we look at this slice in a larger sets in the next two figures, we see the cylinder behavior is very much a local phenomena, hence the need to work in polydiscs in the theorem. These figures were generated

using Mathematica.

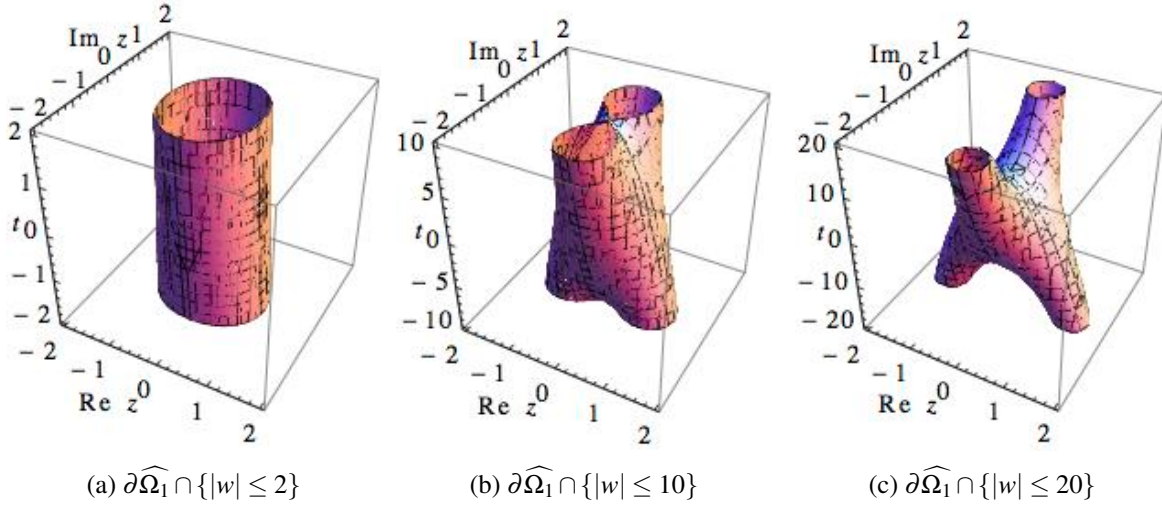


Figure 3.2: Pictures of  $\partial\widehat{\Omega}_1$  near the origin, at different scales.

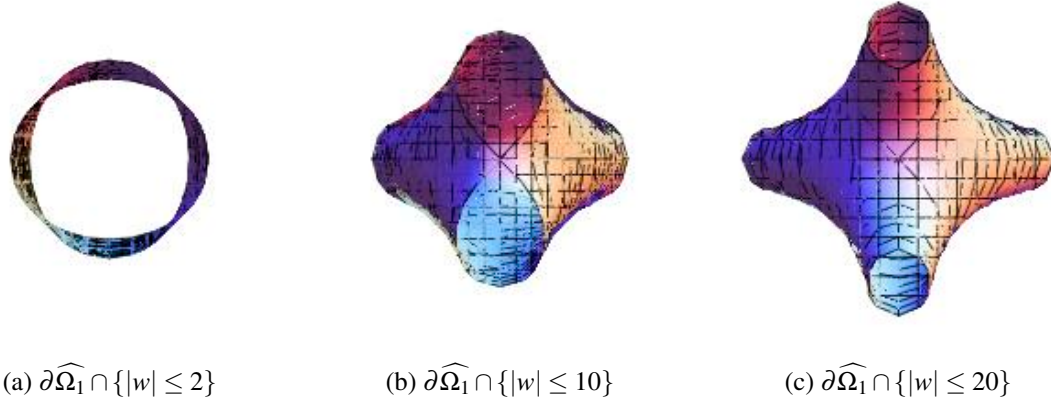


Figure 3.3: Pictures of  $\partial\widehat{\Omega}_1$  near the origin, at different scales, viewed down the  $t$ -axis.

We now prove Lemma 3.5.1. The case  $\ell = 1$  was proved by Globevnik in [Globevnik, 1997]. In the  $\ell = 1$  case the parameter  $M$  plays a minor role. We will extend the lemma to the cases  $\ell \in \mathbb{N}$ , though we will have to keep track of the parameter  $M$ .

### Proof of Lemma 3.5.1.

For clarity's sake the  $\ell = 1$  case will be proven first. Let  $z = re^{i\theta}$ :

$$|z^M + \alpha w| = 1$$



$$|z^M + \alpha w|^2 = 1$$

$$(z^M + \alpha w)(\overline{z^M + \alpha w}) = 1$$

$$\begin{aligned} r^{2M} + \alpha \left( we^{-Mi\theta} + \bar{w}e^{Mi\theta} \right) r^M + \alpha^2 |w|^2 - 1 \\ = r^{2M} + Br^M + C = 0 \end{aligned}$$

where

$$B = \alpha \left( we^{-Mi\theta} + \bar{w}e^{Mi\theta} \right), \quad C = \alpha^2 |w|^2 - 1.$$

The quadratic formula gives:

$$r^M = \frac{-B \pm \sqrt{B^2 - 4C}}{2}$$

With  $\alpha$  small, the discriminant is positive and there are real roots. Choose the positive root ( $r \geq 0$ ).

Note that:

$$\begin{aligned} \frac{\partial B}{\partial \theta} &= iM\alpha \left( -we^{-Mi\theta} + \bar{w}e^{Mi\theta} \right) \\ \frac{\partial B}{\partial w} &= \alpha e^{-Mi\theta}, \quad \frac{\partial B}{\partial \bar{w}} = \alpha e^{Mi\theta} \\ \frac{\partial C}{\partial \theta} &= 0, \quad \frac{\partial C}{\partial w} = \alpha^2 \bar{w}, \quad \frac{\partial C}{\partial \bar{w}} = \alpha^2 w. \end{aligned}$$

and therefore

$$\left| \frac{\partial B}{\partial \theta} \right| \leq 2M\alpha |w|, \quad \left| \frac{\partial B}{\partial \bar{w}} \right| = \left| \frac{\partial B}{\partial w} \right| \leq \alpha |w|, \text{ and } \left| \frac{\partial C}{\partial \bar{w}} \right| = \left| \frac{\partial C}{\partial w} \right| \leq \alpha^2 |w|.$$

Implicitly differentiating

$$r^M = \frac{-B + \sqrt{B^2 - 4C}}{2}$$

gives us:

$$Mr^{M-1} \frac{\partial r}{\partial \theta} = -\frac{\frac{\partial B}{\partial \theta}}{2} + \frac{1}{2\sqrt{B^2 - 4C}} B \frac{\partial B}{\partial \theta}$$

and

$$2r \frac{\partial r}{\partial w} = -\frac{\frac{\partial B}{\partial w}}{2} + \frac{1}{4\beta^2 \sqrt{B^2 - 4C}} \left( 2B \frac{\partial B}{\partial w} - 4 \frac{\partial C}{\partial w} \right)$$

If  $|w| \leq R < \infty$ , then one can choose  $\alpha$  small so that

$$\left| \frac{\partial B}{\partial \theta} \right|, \left| \frac{\partial B}{\partial w} \right|, \left| \frac{\partial B}{\partial \bar{w}} \right|, \left| \frac{\partial C}{\partial w} \right|, \left| \frac{\partial C}{\partial \bar{w}} \right|, |B|$$

are small and  $|4C| = \left| 4(\alpha^2 |w|^2 - 1) \right| \approx 4 > 1$ , then the partial derivatives  $\frac{\partial r}{\partial \theta}$ ,  $\frac{\partial r}{\partial w}$ ,  $\frac{\partial r}{\partial \bar{w}}$  have small modulus. Furthermore:

$$\begin{aligned} r^M &\geq \frac{1}{2} \left( -|B| + \sqrt{|B|^2 - 4(\alpha^2 R^2 - 1)} \right) \geq \frac{1}{2} (-\varepsilon + \sqrt{\varepsilon + 4(1 - \varepsilon)}) \\ &\geq \frac{2\sqrt{1 - \varepsilon} - \varepsilon}{2} = \sqrt{1 - \varepsilon} - \frac{\varepsilon}{2} \end{aligned}$$

and

$$\begin{aligned} r^M &\leq \frac{1}{2} \left( |B| + \sqrt{|B|^2 + 4(\alpha^2 R^2 + 1)} \right) \leq \frac{1}{2} (\varepsilon + \sqrt{\varepsilon + 4(\varepsilon + 1)}) \\ &\leq \frac{2\sqrt{1 + 2\varepsilon} + \varepsilon}{2} = \sqrt{1 + 2\varepsilon} + \frac{\varepsilon}{2} \end{aligned}$$

*Remark 4.* Suppose  $t > 0$  and consider  $\left| \left( \frac{z}{t} \right)^M + \alpha w \right| = 1$ . If the above proof is followed, replacing  $r$  with  $\frac{r}{t}$  then it follows that  $\left| \frac{r}{t} - 1 \right| < \alpha CM^{2\ell}$ , in other words  $\alpha$  can be chosen so that  $r$  uniformly has image values as close as desired to  $t$ . Since  $D^\alpha \frac{r}{t} = \frac{1}{t} D^\alpha r$ , the general case of the lemma applies, i.e.  $\left| \left( \frac{z}{t} \right)^M + \alpha w \right| = 1$  is locally the graph given by  $r$  with  $\|r - t\|_{C^k(\partial\Delta \times R\Delta)} < \alpha CM^{2\ell}$ .

In the general case, our main concern is to keep track of the parameter  $M$ . The rest of the

expression we will be content to just bound. Let

$$2r^M = -B + \sqrt{B^2 - 4C}$$

i.e.

$$r = \sqrt[M]{\frac{-B + \sqrt{B^2 - 4C}}{2}}$$

To simplify notation write

$$r = \sqrt[M]{\frac{\sqrt{4 - (4\alpha^2 |w|^2 - B^2)} - B}{2}} = \sqrt[M]{\sqrt{1 - v} - u}$$

where  $u = B$ , redefine  $\alpha = \frac{\alpha}{2}$  and

$$\begin{aligned} v &= 4\alpha^2 |w|^2 - \alpha^2 \left( w^2 e^{-2Mi\theta} + 2w\bar{w} + \bar{w}^2 e^{i2M\theta} \right) \\ &= \alpha^2 (2|w|^2 - w^2 e^{-2Mi\theta} - \bar{w}^2 e^{i2M\theta}) \end{aligned}$$

Note that  $u, v$  have at most two nonzero derivatives in the variables  $w, \bar{w}$ , and

$$\begin{aligned} \left| \frac{\partial u}{\partial w} \right| &= \left| \frac{\partial u}{\partial \bar{w}} \right| = \alpha \\ \left| \frac{\partial v}{\partial w} \right| &= \left| \alpha^2 (2\bar{w} - 2we^{-2Mi\theta}) \right| \leq 4\alpha^2, \quad \left| \frac{\partial v}{\partial \bar{w}} \right| \leq 4\alpha^2 \\ \left| \frac{\partial^2 v}{\partial w^2} \right| &= \left| \alpha^2 (-2e^{-2Mi\theta}) \right| \leq 2\alpha^2, \quad \left| \frac{\partial^2 v}{\partial \bar{w}^2} \right| \leq 2\alpha^2 \end{aligned}$$

In the variable  $\theta$ , observe:

$$\begin{aligned} \frac{\partial^\ell u}{\partial \theta^\ell} &= M^\ell \alpha \left( (-1)^\ell w e^{-Mi\theta} + \bar{w} e^{Mi\theta} \right) \\ \frac{\partial^\ell v}{\partial \theta^\ell} &= -\alpha^2 (2M)^\ell \left( (-1)^\ell w^2 e^{-2Mi\theta} + \bar{w}^2 e^{2Mi\theta} \right). \end{aligned}$$

Notice that

$$\begin{aligned} \left| \frac{\partial^\ell u}{\partial \theta^\ell} \right| &= M^\ell \alpha \left| (-1)^\ell w e^{-Mi\theta} + \bar{w} e^{Mi\theta} \right| \leq 2\alpha M^\ell |w| \\ \left| \frac{\partial^\ell v}{\partial \theta^\ell} \right| &= \alpha^2 (2M)^\ell \left| (-1)^\ell w^2 e^{-2Mi\theta} + \bar{w}^2 e^{2Mi\theta} \right| \leq 2\alpha^2 (2M)^\ell |w^2|. \end{aligned}$$

Thus for the multi-index  $\eta \in \mathbb{N}^3$ , with  $0 < \alpha < 1$  and for  $\rho = u, v$

$$|D^\eta \rho| \leq \alpha 2^{|\eta|+2} M^{|\eta|} \max\{1, |w^2|\} = \alpha 2^{|\eta|+2} M^{|\eta|} \max\{1, |w^2|\}. \quad (3.5.7)$$

For convenience define  $C(\eta, w) = \alpha 2^{|\eta|} M^{|\eta|+2} \max\{1, |w^2|\}$ .

Using (3.5.7) notice that if  $\beta + \gamma = \eta$ ,

$$C(\beta, w) C(\gamma, w) \leq C(\eta, w) \quad (3.5.8)$$

$$C(\beta, w) \leq C(\eta, w) \quad (3.5.9)$$

We now estimate the modulus of  $D^\eta (\sqrt{1-v} - u)^{\frac{1}{M}}$ , for  $\eta \in \mathbb{N}^3$ . Let

$$f(g(u, v)) = (\sqrt{1-v} - u)^{\frac{1}{M}} = h(u, v)$$

(i.e.  $f(*) = (*)^{\frac{1}{M}}$ ,  $g(u, v) = \sqrt{1-v} - u$ .)

Observing that

$$D^n f(x) = \left( \frac{1}{M} \right) \left( \frac{1}{M} - 1 \right) \dots \left( \frac{1}{M} - n + 1 \right) (x)^{\frac{1}{M} - n}$$

we get the estimate:

$$|D^n f| \leq (n-1)! |x|^{\frac{1}{M} - n}. \quad (3.5.10)$$

We now introduce a notation that will simplify keeping track of our constants.

Let  $B_{m,k}(y_1, \dots, y_{m-k+1})$  denote the Bell polynomial

$$B_{m,k}(y_1, \dots, y_{m-k+1}) = \frac{1}{k!} \sum_{j_1 + \dots + j_k = m, j_i \geq 1} \binom{m}{j_1, \dots, j_k} y_{j_1} \cdots y_{j_k}$$

where  $\binom{m}{j_1, \dots, j_k} = \frac{m!}{j_1! j_2! \cdots j_k!}$  with  $j_1 + j_2 + \cdots + j_k = m$ .

Following [Johnson, 2002], if  $f$  and  $g$  have enough derivatives,

$$\frac{d^m}{dt^m} f(g(t)) = \sum_{k=0}^m f^{(k)}(g(t)) B_{m,k} \left( g'(t), g''(t), \dots, g^{m+k-1}(t) \right) \quad (3.5.11)$$

Applying this formula in our context yields

$$\begin{aligned} \frac{\partial^m}{\partial u^m} f(g_1(u) + g_2(v)) \\ = \sum_{k=0}^m f^{(k)}(g(g_1(u) + g_2(v))) B_{m,k} \left( g'_1(u), g''_1(u), \dots, g_1^{m+k-1}(u) \right) \end{aligned} \quad (3.5.12)$$

Now let  $h(u, v) = f(g_1(u) + g_2(v))$  and let  $u = u(x_1, x_2, x_3), v = v(x_1, x_2, x_3)$ . Then we apply (3.5.11):

$$\begin{aligned} \frac{\partial^n}{\partial x_\ell^n} h(u(x_1, x_2, x_3), v(x_1, x_2, x_3)) \\ = \sum_{j=0}^n \sum_{k=0}^j f^{(k)}(g_1(u) + g_2(v)) B_{m,k} \left( g'_1(u), g''_1(u), \dots, g_1^{j+k-1}(u) \right) \\ \times B_{n,j} \left( u_{x_\ell}, \dots, u_{x_\ell^{n+j-1}} \right) \\ + \sum_{j=0}^n \sum_{k=0}^j f^{(k)}(g_1(u) + g_2(v)) B_{m,k} \left( g'_2(v), g''_2(v), \dots, g_2^{j+k-1}(v) \right) \\ \times B_{n,j} \left( v_{x_\ell}, \dots, v_{x_\ell^{n+j-1}} \right) \end{aligned} \quad (3.5.13)$$

The first partial derivatives of  $g(u, v) = g_1(u) + g_2(v) = \sqrt{1-v} - u$  are respectively:

$$g_u = 1$$

$$g_v = -\frac{1}{2}(1-v)^{-\frac{1}{2}}$$

We see that all mixed partial derivatives  $g_{j,k} = \frac{\partial^{j+k} g}{\partial u^j \partial v^k}$ ,  $j, k \neq 0$  are zero. On the other hand, the derivatives in  $v$  are of the form:

$$\frac{\partial g}{\partial v^k} = -\left(1 - \frac{1}{2}\right) \left(\frac{1}{2}\right) \left(\frac{3}{2}\right) \cdots \left(\frac{1-2k}{2} + 1\right) (1-v)^{\frac{1-2k}{2}}.$$

We can choose  $\alpha$  small so that in the compact set  $R\Delta \times R\Delta$ , both

$$g = \sqrt{1-v} - u, \quad 1-v$$

are arbitrarily close to 1 in modulus. In particular,

$$\frac{1}{2} \leq \left| \sqrt{1-v} - u \right| \leq 2 \quad \text{and} \quad \frac{1}{2} \leq |1-v| \leq 2$$

Thus given  $(j, k) \in \mathbb{N}^2$  one can estimate

$$\left| \frac{\partial^{j+k} g}{\partial u^j \partial v^k} \right| \leq (j+k-1)! 2^{j+k}.$$

Now note that

$$\left| B_{m,k}(y_1, \dots, y_{m-k+1}) \right| \leq B_{m,k}(1, \dots, 1) \max_{j_1 + \dots + j_k = m, j_i \geq 1} \{|y_{j_1} \cdots y_{j_k}|\}$$

i.e.

$$\left| B_{m,k}(g'(t), g''(t), \dots, g^{m+k-1}(t)) \right| \leq B_{m,k}(1, \dots, 1) (m-1)! 2^m$$

For book keeping purposes we will now define Stirling numbers of the second kind. In general,

$B_{m,k}(1,1,\dots,1) = \left\{ \begin{smallmatrix} m \\ k \end{smallmatrix} \right\}$ , where  $\left\{ \begin{smallmatrix} m \\ k \end{smallmatrix} \right\}$  is the  $(m,k)$  Stirling number of the second kind. The Stirling numbers of the second kind  $\left\{ \begin{smallmatrix} m \\ k \end{smallmatrix} \right\}$  can be generated by the recursion relation:

$$\left\{ \begin{smallmatrix} n+1 \\ k \end{smallmatrix} \right\} = k \left\{ \begin{smallmatrix} n \\ k \end{smallmatrix} \right\} + \left\{ \begin{smallmatrix} n \\ k-1 \end{smallmatrix} \right\}$$

For  $k > 0$  with initial conditions

$$\left\{ \begin{smallmatrix} 0 \\ 0 \end{smallmatrix} \right\} = 1 \quad \text{and} \quad \left\{ \begin{smallmatrix} n \\ 0 \end{smallmatrix} \right\} = \left\{ \begin{smallmatrix} 0 \\ n \end{smallmatrix} \right\} = 0.$$

For  $n > 0$ . We return now to the proof of lemma.

Using (3.5.13), for  $\ell \in \{1, 2, 3\}$  and  $n \in \mathbb{N}$ ,

$$\begin{aligned} & \left| \frac{\partial^n}{\partial x_\ell^n} h(u(x_1, x_2, x_3), v(x_1, x_2, x_3)) \right| \\ & \leq \sum_{j=0}^n \sum_{k=0}^j f^{(k)}(g_1(u) + g_2(v)) B_{m,k}(1, \dots, 1) (m-1)! 2^m B_{n,j} \left( u_{x_\ell}, \dots, u_{x_\ell^{n+j-1}} \right) \\ & \quad + \sum_{j=0}^n \sum_{k=0}^j f^{(k)}(g_1(u) + g_2(v)) B_{m,k}(1, \dots, 1) (m-1)! 2^m B_{n,j} \left( v_{x_\ell}, \dots, v_{x_\ell^{n+j-1}} \right) \end{aligned} \quad (3.5.14)$$

or

$$\begin{aligned} & \left| \frac{\partial^n}{\partial x_\ell^n} h(u(x_1, x_2, x_3), v(x_1, x_2, x_3)) \right| \\ & \leq \sum_{j=0}^n \sum_{k=0}^j f^{(k)}(g_1(u) + g_2(v)) \left\{ \begin{smallmatrix} m \\ k \end{smallmatrix} \right\} (m-1)! 2^m B_{n,j} \left( u_{x_\ell}, \dots, u_{x_\ell^{n+j-1}} \right) \\ & \quad + \sum_{j=0}^n \sum_{k=0}^j f^{(k)}(g_1(u) + g_2(v)) \left\{ \begin{smallmatrix} m \\ k \end{smallmatrix} \right\} (m-1)! 2^m B_{n,j} \left( v_{x_\ell}, \dots, v_{x_\ell^{n+j-1}} \right) \end{aligned} \quad (3.5.15)$$

Let  $C(n, w) = \alpha 2^n M^{n+2} \max\{1, |w^2|\}$ . (Since  $C(\eta, w)$  actually only depends on  $|\eta|$ .) Then

(3.5.13) and (3.5.7) on the remaining Bell polynomials above gives:

$$\begin{aligned}
& \left| \frac{\partial^n}{\partial x_\ell^n} h(u(x_1, x_2, x_3), v(x_1, x_2, x_3)) \right| \\
& \leq \sum_{j=0}^n \sum_{k=0}^j f^{(k)}(g_1(u) + g_2(v)) \left\{ \begin{matrix} m \\ k \end{matrix} \right\} (m-1)! 2^m \left\{ \begin{matrix} n \\ j \end{matrix} \right\} C(n, w) \\
& \quad + \sum_{j=0}^n \sum_{k=0}^j f^{(k)}(g_1(u) + g_2(v)) \left\{ \begin{matrix} m \\ k \end{matrix} \right\} (m-1)! 2^m \left\{ \begin{matrix} n \\ j \end{matrix} \right\} C(n, w)
\end{aligned} \tag{3.5.16}$$

now if  $\alpha$  is small so that  $\frac{1}{2} \leq |\sqrt{1-v} - u| \leq 2$ , using (3.5.10):

$$\begin{aligned}
& \left| \frac{\partial^n}{\partial x_\ell^n} h(u(x_1, x_2, x_3), v(x_1, x_2, x_3)) \right| \\
& \leq \sum_{j=0}^n \sum_{k=0}^j (k-1)! 2^k \left\{ \begin{matrix} m \\ k \end{matrix} \right\} (m-1)! 2^m \left\{ \begin{matrix} n \\ j \end{matrix} \right\} C(n, w) \\
& \quad + \sum_{j=0}^n \sum_{k=0}^j (k-1)! 2^k \left\{ \begin{matrix} m \\ k \end{matrix} \right\} (m-1)! 2^m \left\{ \begin{matrix} n \\ j \end{matrix} \right\} C(n, w) \\
& \leq K(n) C(n, w)
\end{aligned} \tag{3.5.17}$$

For some constant  $K$  depending on  $n$ . Fixing a positive integer  $m$ , the factor  $\alpha$  in (3.5.7):

$$C(\eta, w) = \alpha 2^{|\eta|+2} M^{|\eta|} \max\{1, |w^2|\}$$

can be chosen to make this last estimate as small as desired for  $(z, w) \in R\Delta \times R\Delta$  and  $|\eta| \leq m$ .

□



# Chapter 4

## Fatou-Bieberbach Domain with Gevrey

### Class Boundary

In many applications, smoothness is a desirable property of a function. For example, we may consider the Schwartz class functions, and their role in the theory of distributions. Stensönes in [Stensönes, 1997] demonstrated the existence of a Fatou-Bieberbach domain in  $\mathbb{C}^2$  with  $C^\infty$ -smooth boundary. The Gevrey classes of functions extend the notion of smoothness of a function beyond  $C^\infty$ , by requiring not only that derivatives exist, but that their modulus is controlled by particular bounds on compact sets. The Gevrey classes allow us to consider grades of smoothness from the class  $G^\infty(\Omega)$ , i.e. the set of  $C^\infty$ -smooth functions on  $\Omega$ , up to the class  $G^1(\Omega)$ , i.e. the analytic functions on  $\Omega$ .

We will use the Gevrey classes to extend the boundary smoothness of Stensönes' domain  $\Omega$  because she created a defining function  $p : \mathbb{C}^2 \rightarrow \mathbb{R}$ ,  $\Omega = \{z \in \mathbb{C}^2 : p(z) < 0\}$  and proved estimates on the (real) derivatives of  $p$  that allow for relatively direct computation of the estimates that define the Gevrey Classes.

#### 4.1 Stensönes' Example

**Theorem 6.** [Stensönes, 1997] *There exists a Fatou-Bieberbach domain in  $\mathbb{C}^2$  with  $C^\infty$ -boundary.*

## Stensönes' Method

Stensönes' used a sequence of polynomial automorphisms  $\{H_j\}$  where:

$$G_n(z, w) = H_n \circ H_{n-1} \circ \cdots \circ H_1(z, w)$$

and

$$\Omega = \{(z, w) \in \mathbb{C}^2 : \lim_{n \rightarrow \infty} G_n(z, w) = 0\}.$$

Stensönes' used a varying sequence of automorphisms and what we will call the "push-out" method to construct her domain. The particular automorphisms she employed are called polynomial shears, which we now define. A polynomial shear on  $\mathbb{C}^2$  is an automorphism  $H$  of  $\mathbb{C}^2$  of the form

$$H : (z_1, z_2) \mapsto (z_1, z_2 + p(z_1))$$

or

$$H : (z_1, z_2) \mapsto (z_1 + p(z_2), z_2)$$

for  $p$  a polynomial with coefficients in  $\mathbb{C}$ . Stensönes' used the polynomial shears

$$H_k(z, w) = \begin{cases} \left( z, w + T_k \left( \frac{z}{T_{k-1}} \right)^{N_k} \right) & \text{for } k \text{ odd} \\ \left( z + T_k \left( \frac{w}{T_{k-1}} \right)^{N_k}, w \right) & \text{for } k \text{ even} \end{cases}$$

where  $N_k$  is a sequence of positive integers and  $T_k = e^{3\sqrt{N_k}}$ .

Let  $(z_n, w_n) = G_n(z, w) = H_n \circ H_{n-1} \circ \cdots \circ H_1(z, w)$ . Define

$$\Omega_k = \begin{cases} \{(z, w) \in \mathbb{C}^2 : \left| \frac{w_k(z, w)}{T_k} \right| < 1\} & \text{if } k \text{ is odd} \\ \{(z, w) \in \mathbb{C}^2 : \left| \frac{z_k(z, w)}{T_k} \right| < 1\} & \text{if } k \text{ is even} \end{cases}$$

Choose a sequence of positive integers  $R_k, R_k \rightarrow \infty$ . Then the  $N_k$ 's are chosen large so that in

the ball  $B(0, R_k) \subset \mathbb{C}^2$ ,  $\partial\Omega_k \approx \partial\Omega_{k+1}$ . (There exist a sequence of shrinking open neighborhoods of  $U_k$  of  $\partial\Omega_k$  such that in the ball  $B(0, R_k)$ ,  $\partial\Omega_{k+1} \subset U_k$ .)

More explicitly, suppose  $k$  is odd. Then we let

$$\begin{aligned}\Omega_k &= \left\{ \left| \frac{w_k(z, w)}{T_k} \right| < 1 \right\} = \left\{ \left| \frac{w_{k-1} + T_k \left( \frac{z_{k-1}}{T_{k-1}} \right)^{N_k}}{T_k} \right| < 1 \right\} \\ &= \left\{ \left| \frac{w_{k-1}}{T_k} + \left( \frac{z_{k-1}}{T_{k-1}} \right)^{N_k} \right| < 1 \right\}.\end{aligned}$$

The main idea is that if the numbers  $N_k$  (and hence  $T_k$ ) are chosen large enough, then the domains  $\Omega_k$  converge in  $k$  to a smoothly-bounded domain  $\Omega$ . For example, looking at the equality  $\Omega_k = \left\{ \left| \frac{w_{k-1}}{T_k} + \left( \frac{z_{k-1}}{T_{k-1}} \right)^{N_k} \right| < 1 \right\}$  we see that if  $T_k$  is large enough, locally this set will look like the set  $\left| \left( \frac{z_{k-1}}{T_{k-1}} \right)^{N_k} \right| < 1$ , whose boundary is just a cylinder in the coordinates  $(z_{k-1}, w_{k-1})$ , and so locally  $\partial\Omega_k \approx \partial\Omega_{k+1}$ .

## 4.2 Gevrey Class

Let  $\Omega \subset \mathbb{R}^n$  and  $s \geq 1$ . The Gevrey Class  $G^s(\Omega)$  of index  $s$  is the set of  $f \in C^\infty(\Omega)$  such that for every compact  $K \subset \Omega$  there exists a  $C = C_{f,K} > 0$  for which:

$$\max_{x \in K} |\partial^\alpha f(x)| \leq C^{|\alpha|+1} (|\alpha|!)^s$$

where  $\alpha \in \mathbb{Z}_+^n$ ,  $|\alpha| = \alpha_1 + \dots + \alpha_n$ . In order to apply this definition to our domain in  $\mathbb{C}^2$  we identify  $\mathbb{C}^2$  with  $\mathbb{R}^4$  in the standard way. For any complex number  $z$  let the real part of  $z$  be denoted  $\operatorname{Re} z$  and let the complex part of  $z$  be denoted  $\operatorname{Im} z$ , so  $z = \operatorname{Re} z + i \operatorname{Im} z$ . If  $(z_1, z_2) \in \mathbb{C}^2$ , we write  $x_1 = \operatorname{Re} z_1, y_1 = \operatorname{Im} z_1$ , and  $x_2 = \operatorname{Re} z_2, y_2 = \operatorname{Im} z_2$ , and we have the isomorphism  $(z_1, z_2) \mapsto (x_1, y_1, x_2, y_2)$ .

Let us consider some examples. The function

$$f(x) = \begin{cases} \exp(-\frac{1}{x}) & x > 0 \\ 0 & x \leq 0 \end{cases}$$

is of Gevrey Class 2.

An example from differential equations is given by any weak solution  $u$  to the heat equation

$$\frac{\partial u}{\partial t} = \sum_{j=1}^n \frac{\partial u}{\partial x_j}.$$

Such a solution  $u$  satisfies:  $u \in G^s(\Omega)$  for  $s \geq 2$ , while  $u \notin G^s(\Omega)$  for  $1 \leq s < 2$ .

### 4.3 A Gevrey Class-Smooth Fatou-Bieberbach Domain

We will now show that the polynomial shears in Stensönes' domain can be chosen so that the resulting domain has Gevrey class 4 boundary. Suppose, as shown in [Stensönes, 1997], that there exists a sequence of balls

$$B(0, R_n) = \{(z, w) \in \mathbb{C}^2 : \|(z, w)\| < R_n\}, \quad R_n \rightarrow \infty$$

such that  $\Omega \cap B(0, R_n)$  is given by a defining function  $r_n$  of the form  $r_n = |P_n(z, w)| + \sum_{j>n} g_j$ , where  $P_n$  is a polynomial and the  $g_j$ 's satisfy the inequality:

$$|D^\alpha g_j| \leq 2^{m+1} (m!)^2 N_i^{2m} e^{-2N_j} \quad (4.3.1)$$

where  $|\alpha| = m$  and the  $N_j$ 's are a sequence of integers.

**Theorem 7 (C).** *Let  $\Omega$  be the domain defined in [Stensönes, 1997]. Then there are defining functions  $r_n$  satisfying (4.3.1) such that the domain  $\Omega$  has class  $G^4$  smooth boundary.*

**Proof of Theorem 7.**

Let  $\Omega$  be the domain defined in [Stensönes, 1997], with defining functions  $r_n$  satisfying (4.3.1), as in [Stensönes, 1997]. On any compact set  $K \subset \Omega$ , one can find an  $n(K) = n \in \mathbb{N}$  and  $C(K) = C \in \mathbb{R}$  such that  $K \subset B(0, R_n)$  and:

$$|D^\alpha r| \leq \sum_{j=1}^{\infty} |D^\alpha g_j| + C \leq C^{m+1} (m!)^2 \sum_{j=1}^{\infty} N_j^{2m} e^{-2N_j}$$

holds on the set  $K$ . We will prove the inequality

$$C^{m+1} (m!)^2 \sum_{j=1}^{\infty} N_j^{2m} e^{-2N_j} \leq C^{m+1} (m!)^4.$$

Split this series into the two parts:

$$\sum_{j=1}^{\infty} N_j^{2m} e^{-2N_j} = \sum_{N_j \leq m} N_j^{2m} e^{-2N_j} + \sum_{N_j > m} N_j^{2m} e^{-2N_j} = I + II$$

and estimate these two parts separately.

Note that the function  $f : \mathbb{R}_+ \rightarrow \mathbb{R}$  given by

$$f(x) = x^{2m} e^{-2x}$$

Has a unique maximum at  $x = m$  and decreases monotonically for  $x > m$ . Hence, for  $j \in \mathbb{N}$ :

$$N_j^{2m} e^{-2N_j} \leq \frac{m^{2m}}{e^{2m}}$$

so one obtains the estimate:  $\sum_{N_j \leq m} N_j^{2m} e^{-2N_j} \leq m \frac{m^{2m}}{e^{2m}}.$

For the other term, it may be assumed that  $N_j$  grows at least as fast as  $2^j$ . Assuming this,

$$II = \sum_{N_j > m} N_j^{2m} e^{-2N_j} \leq \sum_{j=m+1}^{\infty} (2^j)^{2m} e^{-2^{j+1}}$$

and comparing successive terms in the sum  $\sum_{j=m+1}^{\infty} (2^j)^{2m} e^{-2^{j+1}}$ ,

$$\frac{e^{2m \ln(2^{j+1}) - 2(2^{j+1})}}{e^{2m \ln(2^j) - 2(2^j)}} = e^{2m \ln(2) + 2^{j+1} - (2^{j+2})} = e^{2m \ln(2) - 2^{j+1}}$$

and  $2m \ln(2) - 2^{j+1} \leq -\frac{1}{2}$  when  $j \geq m$ .

So

$$\begin{aligned} \sum_{j=1}^{\infty} N_j^{2m} e^{-2N_j} &\leq \sum_{j \leq m} \frac{m^{2m}}{e^{2m}} + \sum_{j > m} \frac{(2^j)^{2m}}{e^{2(2^j)}} \\ &\leq m \frac{m^{2m}}{e^{2m}} + \frac{m^{2m}}{e^{2m}} \sum_{j=2}^{\infty} e^{-\frac{j}{2}} \\ &\leq 2m \frac{m^{2m}}{e^{2m}} \end{aligned}$$

Finally, using Stirling's approximation  $\sqrt{2\pi m} \left(\frac{m}{e}\right)^m \approx m!$ , and in particular,

$$\sqrt{2\pi m} \left(\frac{m}{e}\right)^m \leq m! \leq \frac{e}{\sqrt{2\pi}} \sqrt{2\pi m} \left(\frac{m}{e}\right)^m,$$

we conclude

$$|D^\alpha r| \leq C^{m+1} (m!)^2 2m \frac{m^{2m}}{e^{2m}} = C^{m+1} (m!)^2 2m \left(\frac{m}{e}\right)^{2m} \leq C^{m+1} (m!)^4.$$

□

*Remark 5.* At first it may seem we should be able to greatly improve this result. After all, the constants  $N_j$  must go to infinity and we can choose them to grow as quickly as we like. However, If we look back at the sum

$$\sum_{j=1}^{\infty} N_j^{2m} e^{-2N_j} = \sum_{N_j \leq m} N_j^{2m} e^{-2N_j} + \sum_{N_j > m} N_j^{2m} e^{-2N_j} = I + II$$

we see that this freedom to choose natural numbers  $N_j$  does not allow us to avoid the scenario

$m = N_j$ . Fix  $J \in \mathbb{N}$  and let  $m_J = N_J$ . Then even the under estimate

$$\sum_{j=1}^{\infty} N_j^{2m_J} e^{-2N_j} = I + II > I \geq \frac{m_J^{2m_J}}{e^{2m_J}}$$

Again gives us an in equality of the form

$$|D^\alpha r| \leq C^{m_J+1} (m_J!)^4. \quad (4.3.2)$$

Since there are infinitely many orders of derivatives  $m_j$  for which we can make the estimate (4.3.2), we cannot get any help from the choice of the constant  $C$ .

# Chapter 5

## Short- $\mathbb{C}^2$ Domains with Local $C^\ell$ -Smooth Boundary

### 5.1 Fornæss' Construction of Short- $\mathbb{C}^2$ Domains

In the previous chapters we have studied domains  $\Omega \subset \mathbb{C}^2$  of the form:

$$\Omega_1 \subset \Omega_2 \subset \Omega_3 \subset \cdots \subset \Omega_n \subset \cdots, \quad \Omega = \bigcup \Omega_n$$

where each  $\Omega_j$  is biholomorphic to the unit polydisc, and the union is a Fatou-Bieberbach domain. In this chapter we consider another domain arising from an increasing union of domains biholomorphic to the polydisc, called a "short- $\mathbb{C}^2$ ". Assume  $\Omega$  is given by an increasing union of domains  $\Omega_n$ , and each  $\Omega_n$  is biholomorphic to the unit ball in  $\mathbb{C}^2$ . If  $\Omega$  is not biholomorphic to  $\mathbb{C}^2$ , then it is possible that either  $\Omega$  is biholomorphic to the unit ball or the unit disc cross  $\mathbb{C}$ . Another possibility is that  $\Omega$  is a short- $\mathbb{C}^2$ . More explicitly, Fornæss and Sibony in 1981 [Fornæss & Sibony, 1981] showed that if each  $\Omega_j$  is biholomorphic to the unit ball in  $\mathbb{C}^2$  and the Kobayashi metric of  $\Omega$  is not identically zero, then  $\Omega$  is biholomorphic to the unit ball or to  $\Delta \times \mathbb{C}$ . A Short- $\mathbb{C}^2$  domain  $\Omega$  is



a subset of  $\mathbb{C}^2$  of the form

$$\Omega_1 \subset \Omega_2 \subset \Omega_3 \subset \cdots \subset \Omega_n \subset \cdots, \quad \Omega = \bigcup \Omega_n$$

where each  $\Omega_j$  is biholomorphic to the unit ball,  $\Omega$  has identically zero infinitesimal Kobayashi metric, and  $\Omega$  supports a bounded, nonconstant, plurisubharmonic function. The existence of this bounded, nonconstant plurisubharmonic function guarantees that  $\Omega$  is not biholomorphic to  $\mathbb{C}^2$ .

*Remark 6.* The infinitesimal Kobayashi metric of  $\Omega$  vanishes identically if and only if for all  $p \in \Omega$  and any tangent vector  $\xi$  to  $\Omega$  at  $p$  and for any  $R > 0$ , there exists a holomorphic map  $f : \Delta \rightarrow \Omega$  such that  $f(0) = p$  and  $f'(0) = R\xi$ . We can think of the infinitesimal Kobayashi metric of  $\Omega$  as measuring how large a copy of the unit disc can be embedded in  $\Omega$ . For more on the infinitesimal Kobayashi metric, see Krantz's enjoyable survey [Krantz, 2008].

The following Theorem of Fornæss allows one to construct short- $\mathbb{C}^2$  domains. Similarly one can construct short  $\mathbb{C}^k$ -domains, for  $k > 2$ . We begin with some definitions that will clarify the statement of the theorem.

Let  $d \geq 2$  be an integer. For  $\eta > 0$ , let  $\text{Aut}_{d,\eta}$  be the set of polynomial automorphisms  $F$  of  $\mathbb{C}^2$  of the form:

$$F(z_1, z_2, \dots, z_k) = (z_1^d + P_1(z_1, \dots, z_k), P_2(z_1, \dots, z_k))$$

where degree  $P_i \leq d - 1$  and each coefficient is of modulus  $\leq \eta$ , for each  $i$ ,  $i = 1, 2$ .

Let  $F_j \in \text{Aut}_{d,\eta_j}$ ,  $j = 1, 2, \dots, n$  and define

$$F(n) = F_n \circ F_{n-1} \circ \cdots \circ F_1$$

$$\Omega = \{z \in \mathbb{C}^2 \text{ such that } \lim_{n \rightarrow \infty} F(n)(z) = 0\}.$$

**Theorem 8.** [Fornæss, 2004]

If  $\eta_n = a_n^{d^n}$  and  $1 > a_1 \geq a_2 \geq \dots \lim_{n \rightarrow \infty} a_n = a_\infty \geq 0$  then:

1.  $\Omega$  is a nonempty, open, connected set in  $\mathbb{C}^2$

2.  $\Omega = \cup_{j=1}^{\infty} \Omega_j \supset \dots \supset \Omega_l \dots \supset \Omega_1$ . Each  $\Omega_j$  is biholomorphic to the unit ball in  $\mathbb{C}^2$ .
3. The infinitesimal Kobayashi metric of  $\Omega$  vanishes identically.
4. There is a plurisubharmonic function  $\Psi : \mathbb{C}^2 \rightarrow [\log a_{\infty}, \infty)$  such that  $\Omega = \{\Psi < 0\}$  and  $\Psi$  is nonconstant on  $\Omega$ .

Under the hypotheses of Theorem 8, the resulting domain  $\Omega$  is a short- $\mathbb{C}^2$ . The existence of a bounded plurisubharmonic function on  $\Omega$  shows that  $\Omega$  is not biholomorphic to  $\mathbb{C}^2$ . First note that if  $\Omega \subset \mathbb{C}^n$  and  $D \subset \mathbb{C}^m$ , and  $F : D \rightarrow \Omega$  is holomorphic, then if  $u$  is plurisubharmonic on  $\Omega$  then  $u \circ F$  is plurisubharmonic on  $D$  (see [Range, 1986] pg 90 for a proof). Suppose  $\Psi$  is a non-constant and bounded on  $\Omega$ , as above. If  $\Omega$  were biholomorphic to  $\mathbb{C}^2$  (say  $F : \Omega \rightarrow \mathbb{C}^2$ ,  $F(\Omega) = \mathbb{C}^2$  and  $F$  is biholomorphic) then  $\Psi \circ F^{-1}$  would be a non-constant bounded plurisubharmonic function on  $\mathbb{C}^2$ , which cannot be.

To see that  $\Psi \circ F^{-1}$  cannot be a bounded plurisubharmonic function on  $\mathbb{C}^2$ , suppose  $\Psi \circ F^{-1}$  is restricted to any complex line in  $\mathbb{C}^2$ , take some line through the origin. On such a line, let  $v_{\varepsilon} = u(z) - \varepsilon \log |z|$ . Note that on the set  $|z| > 1$ , we have  $v_{\varepsilon} < u$ . Since  $v_{\varepsilon} \rightarrow -\infty$  as  $|z| \rightarrow \infty$ , we have  $\sup_{|z| > 1} v_{\varepsilon} = \max_{\partial \Delta} v_{\varepsilon} = \max_{\partial \Delta} u = \max_{\bar{\Delta}} u$ . Thus on the set  $|z| > 1$ ,  $u(z) = v_{\varepsilon}(z) + \varepsilon \log |z| \leq \max_{\bar{\Delta}} u + \varepsilon \log |z|$ . Hence  $u(z) \leq \max_{\bar{\Delta}} u$  and taking the supremum over  $z$  gives  $\sup u(z) = \max_{\bar{\Delta}} u$ . Therefore the maximum is attained and so by the maximum principle,  $\Psi \circ F^{-1}$  must be constant on said line, hence on all lines through the origin, hence identically constant, a contradiction.

The idea of the proof of Theorem 8 is to pick a constant  $0 < c < 1$  and write  $\Omega$  as a union of some sets  $\{(z, w) \in \mathbb{C}^2 : F(n)(z, w) \in c\Delta \times c\Delta\}$  and then use a Green function construction for the map  $\Psi$ . The proof shows that  $\Omega$  can be written as an increasing union of sets biholomorphic to the ball (or polydisc), and  $\Omega$  is a short  $\mathbb{C}^2$ . The proof does not address the smoothness of the boundary, and we will show that one can choose the coefficients of the  $F_n$  so that the maps satisfy the hypothesis of Theorem 8 and yield a smoothly bounded short- $\mathbb{C}^2$ .

Using Theorem 8, we see that a simple example of a short  $\mathbb{C}^2$  domain is given by the basin of

attraction at the origin for the sequence of automorphisms  $F_n$  where:

$$F_n(z, w) = \left( z^2 + \left( \frac{1}{2} \right)^{2^n} w, \left( \frac{1}{2} \right)^{2^n} z \right).$$

In the same paper, [Fornæss, 2004], Fornæss also proves Theorem 9, which we present below. Theorem 9 shows that if the coefficients  $a_n$  of the maps  $F_n$  decay slower than in Theorem 8, the resulting domain can be Fatou-Bieberbach domain.

**Theorem 9.** [Fornæss, 2004] *Let  $F_n(z, w) = (z^2 + a_n w, a_n z)$ . Suppose that  $0 < |a_n| < c < 1$  and  $|a_{n+1}| \geq |a_n|^t$  for some  $1 < t < 2$ . Then the basin of attraction of 0 is biholomorphic to  $\mathbb{C}^2$ .*

Notice that  $t \geq 2$  in the previous theorem that gives rise to a short- $\mathbb{C}^2$ .

## 5.2 Short $\mathbb{C}^2$ with local $C^\ell$ -boundary

In [Fornæss, 2004], Fornæss did not study boundary smoothness of Short- $\mathbb{C}^2$  domains. We will now demonstrate the existence of a short- $\mathbb{C}^2$  domain whose intersection with a polydisc about the origin will be a small,  $C^\ell$ -smooth perturbation of a cylinder.

**Theorem 10 (C).** *Given  $\ell \in \mathbb{N}$  and  $R > 0$ , there exists a short- $\mathbb{C}^2$  domain  $\Omega \subset \mathbb{C}^2$  such that the set*

$$\partial\Omega \cap (R\Delta \times R\Delta)$$

*is an arbitrarily small  $C^\ell$ -smooth perturbation of the set  $\partial\Delta \times R\Delta$ .*

The domain  $\Omega$  will be the basin of attraction of a sequence of shears. The key to the proof is thoughtful choice of the coefficients of the shears. The heuristic of the argument is based on the following geometric observation: Suppose we choose a polydisc in  $\mathbb{C}^2$  of fixed radius about the origin, and consider the set  $\{(z, w) \in \mathbb{C}^2 : |z^2| = 1\}$  in this polydisc. We see a cylinder. If however, we are given a non-zero function  $f : \mathbb{C} \rightarrow \mathbb{C}$ , and we consider the set  $\{(z, w) \in \mathbb{C}^2 : |z^2 + f(w)| = 1\}$  in the same polydisc, we will longer see a cylinder. However, if we choose a constant  $\alpha$  small

and use it to scale down  $f$ , we can dampen the effect of the  $f(w)$  term and looking at the set  $\{(z, w) \in \mathbb{C}^2 : |z^2 + \alpha f(w)| = 1\}$  in the polydisc, we may again see a set that looks, more or less, like a cylinder.

To prove Theorem 10 we first prove theorem lemma 5.3.1 which allows us to write our short- $\mathbb{C}^2$  domain  $\Omega$  as a union of preimages (under compositions of shears) of polydiscs. We then describe how we describe the boundaries of these primages of polydiscs using parameterizations of cylinders. These local coordinates live in a Banach Space, and we will construct the parameterizations so that they form a Cauchy sequence in this Banach Space. Next we will develop two lemmas (lemma 5.3.4 and lemma 3.5.2) for controlling the properties of these local coordinates. Finally we will use a limit process to complete the proof of Theorem 10.

We will construct a short- $\mathbb{C}^2$  domain  $\Omega$  with local  $C^\ell$ -boundary, using smoothness tools similar to those of chapter 4. We will apply the shears

$$F_n(z, w) = (z^2 + \alpha_n w, \alpha_n z)$$

where the constants  $\alpha_n$  will satisfy:

$$0 < \alpha_1 < 1, \quad \alpha_{n+1} \leq \alpha_n^{2^n} \text{ for } n \geq 1. \quad (5.2.1)$$

Given  $\frac{1}{2} > \varepsilon > 0, R > 0$ , we will obtain  $\Omega \cap \{(z, w) \in \mathbb{C}^2 : |z| < R, |w| < R\}$  as the limit of approximating domains  $\Omega_n \cap \{(z, w) \in \mathbb{C}^2 : |z| < R, |w| < R\}$ , where each approximating domain will be an  $\varepsilon$ -small  $C^\ell$  perturbation of the set  $\{(z, w) \in \mathbb{C}^2 : |z| < 1, |w| < R\}$ . First we will show that the  $\Omega_n$ s converge to a short- $\mathbb{C}^2$  domain  $\Omega$ . Afterwards we will show that the domain  $\Omega$  has the desired boundary smoothness.

### 5.3 Short $\mathbb{C}^2$ as a Limit of Polydisc $\Omega_n$ 's

While short- $\mathbb{C}^2$  are given by an increasing union of domains biholomorphic to the unit disc, it will be more convenient here to work with a union polydiscs, because describing the boundary of a polydisc is better suited for our purposes. Let  $F(n)(z, w) = F_n \circ F_{n-1} \circ \cdots \circ F_1(z, w)$ . We denote the components  $F(n)(z, w)$  by  $(z_n, w_n)$ , i.e.  $F(n)(z, w) = (z_n(z, w), w_n(z, w))$ . Define  $\Omega_n = \{(z, w) \in \mathbb{C}^2 : |z_n(z, w)| < 1, |w_n(z, w)| < 1\}$ . For each  $n \in \mathbb{N}$ ,  $(z, w) \mapsto (z_n(z, w), w_n(z, w))$  is an automorphism of  $\mathbb{C}^2$ , therefore the functions  $|z_n(z, w)|$  and  $|w_n(z, w)|$  are continuous. Let  $R > 0$ . Then the set  $R\Delta \times R\Delta$  is bounded, and therefore one can choose  $\alpha_n$  satisfying (5.2.1) so that in the set  $R\Delta \times R\Delta$ ,  $|w_{n+1}| = |\alpha_n z_n| = |\alpha_n| |z_n| < 1$ . As a result,

$$\partial\Omega_n \cap (R\Delta \times R\Delta) = \{(z, w) \in R\Delta \times R\Delta : |z_n(z, w)| = 1\}. \quad (5.3.1)$$

Notice that it is easy to check that (5.3.1) is true:

$\partial\Omega_n \cap (R\Delta \times R\Delta) = \{(z, w) \in \mathbb{C}^2 : |z_n(z, w)| = 1 \text{ and } |w_n(z, w)| \leq 1, \text{ or } |w_n(z, w)| = 1, |z_n(z, w)| \leq 1\} \cap (R\Delta \times R\Delta)$ . By the assumption on  $\alpha_n$ , we know that  $|w_n(z, w)| < 1$  on  $R\Delta \times R\Delta$ . Therefore  $\partial\Omega_n \cap (R\Delta \times R\Delta) = \{(z, w) \in \mathbb{C}^2 : |z_n(z, w)| = 1 \text{ and } |w_n(z, w)| \leq 1\} \cap (R\Delta \times R\Delta)$ . Therefore if  $(z, w) \in \partial\Omega_n \cap (R\Delta \times R\Delta)$ , then  $|z_n(z, w)| = 1$  and  $(z, w) \in R\Delta \times R\Delta$ , hence  $(z, w) \in \{(z, w) \in R\Delta \times R\Delta : |z_n(z, w)| = 1\}$ .

On the other hand, if  $(z, w) \in \{(z, w) \in R\Delta \times R\Delta : |z_n(z, w)| = 1\}$ , then  $|z_n(z, w)| = 1$  and  $(z, w) \in R\Delta \times R\Delta$ , and it is also true that  $|w_n(z, w)| \leq 1$ , and therefore  $(z, w) \in \partial\Omega_n \cap (R\Delta \times R\Delta)$ .

The point is, we will obtain a short- $\mathbb{C}^2$  domain  $\Omega$  as a union of sets  $\Omega_n$  where each  $\Omega_n$  is a biholomorphic image of a polydisc, such that in the polydisc  $R\Delta \times R\Delta$  we can describe the boundary of each  $\Omega_n$  as an intersection of  $R\Delta \times R\Delta$  with a level set of the form  $\{(z, w) \in \mathbb{C}^2 : |z_n(z, w)| = 1\}$ .

We now present a lemma that allows us to see that the coefficients  $\alpha_n$  of the shears  $F_n$  can be chosen so that the sets  $\Omega_n \cap (R\Delta \times R\Delta)$  converge to  $\Omega \cap (R\Delta \times R\Delta)$ , where  $\Omega = \{(z, w) \in \mathbb{C}^2 : F(n)(z, w) \rightarrow 0\}$  is a short- $\mathbb{C}^2$ . In order to state the lemma we define, for  $n \in \mathbb{N}$ , and  $0 < c < 1$ ,  $\Omega_{n,c} = \{(z, w) \in \mathbb{C}^2 : |z_n| < c, |w_n| < c\}$ . For each  $n$ ,  $\Omega_{n,c}$  is a subset of  $\Omega_{n,1}$  and has the property

that for any appropriate choice of  $\alpha_m, m \geq n$ , we can guarantee that  $\Omega_{m,c} \subset \Omega$ . To illustrate our approximation process we will use  $\varepsilon$ -tubes, defined for any set  $K \subset \mathbb{C}^2$ , and  $\varepsilon > 0$  to be the union of open balls:  $\bigcup_{(z,w) \in K} B_\varepsilon((z,w))$ .

**Lemma 5.3.1.** *Let  $\Omega = \{(z,w) \in \mathbb{C}^2 : F(n)(z,w) \rightarrow 0\}$  and let there be a sequence of positive real numbers  $\{\varepsilon_n\}, \varepsilon_n \searrow 0$ . Given  $n \in \mathbb{N}$ , assume that there exists a positive real number  $\tilde{\alpha}_n$  such that if  $0 < \alpha_n < \tilde{\alpha}_n$ ,*

$$\partial\Omega_n \cap \overline{(R\Delta \times R\Delta)} = \{(z,w) \in \mathbb{C}^2 : |z_n(z,w)| = 1\} \cap \overline{(R\Delta \times R\Delta)}. \quad (5.3.2)$$

$$\partial\Omega_{n-1} \cap \overline{(R\Delta \times R\Delta)} \text{ is in an } \varepsilon_n\text{-tube of } \partial\Omega_n \cap \overline{(R\Delta \times R\Delta)}. \quad (5.3.3)$$

*Then we can choose the coefficients  $\alpha_m, m \geq n$  such that there exists a real number  $c_n, 0 < c_n < 1$  such that  $\Omega_{m,c_n} \subset \Omega$  and  $\partial\Omega_n \cap (R\Delta \times R\Delta)$  is in an  $\varepsilon_n$ -tube of  $\partial\Omega_{n,c_n} \cap (R\Delta \times R\Delta)$ .*

The proof of lemma 5.3.1 will rely on two lemmas, lemma 5.3.2 and lemma 5.3.3. Lemma 5.3.2 gives us upper bound conditions on the coefficients  $\alpha_n$  so that we can ensure the existence of a sequence of  $\Omega_{n,c_n}$ 's so that locally, for each  $n \in \mathbb{N}$ ,  $\Omega_{n,c_n}$  fits into a prescribed neighborhood of  $\Omega_n$ . Lemma 5.3.3 then allows us to find upper bound and decay conditions on the  $\alpha_n$  so that for a prescribed sequence  $\{c_n\}$ , we can guarantee that  $\Omega_{n,c_n}$  is in  $\Omega = \{(z,w) : F(n)(z,w) \rightarrow 0\}$ .

**Lemma 5.3.2.** *Given  $\varepsilon > 0, n \in \mathbb{N}$  and  $K_n$  a compact subset of  $\mathbb{C}^2$  and shears  $F_1, \dots, F_n$  satisfying the hypotheses of Theorem 8 such that  $\partial\Omega_{n,1} \cap K_n = \{(z,w) : |z_n| = 1\} \cap K_n$ , we can find  $0 < c_{n+1} < 1$ ,  $c_{n+1} = c_{n+1}(\varepsilon)$  and  $\alpha'_{n+1} > 0$  such that for all  $0 < \alpha_{n+1} < \alpha'_{n+1}$ ,  $\partial\Omega_{n+1,c_{n+1}} \cap K_n$  is in an- $\varepsilon$ -tube of  $\partial\Omega_{n+1,1} \cap K_n$ .*

**Lemma 5.3.3.** *Given  $N \in \mathbb{N}$ ,  $0 < c < 1$ ,  $0 < a_1 < 1$  and  $\alpha_n = a_1^{b_n}$  for  $n \in \mathbb{N}$ , we can find  $b'_N \in \mathbb{R}, b'_N = b'_{N,c}$  such that if  $b_N \geq b'_N$  and  $b_{n+1} \geq 2b_n$  for  $n \geq N$ , and  $\alpha_n$  for all  $n \in \mathbb{N}$  satisfy the hypotheses of Theorem 8, then  $\Omega_{n,c} \subset \Omega$  for  $n \geq N$ .*

We will use lemma 5.3.2 and lemma 5.3.3 in chapter six and for this reason they are written in more generality than we need to prove lemma 5.3.1. Lemma 5.3.3 and its proof are modified

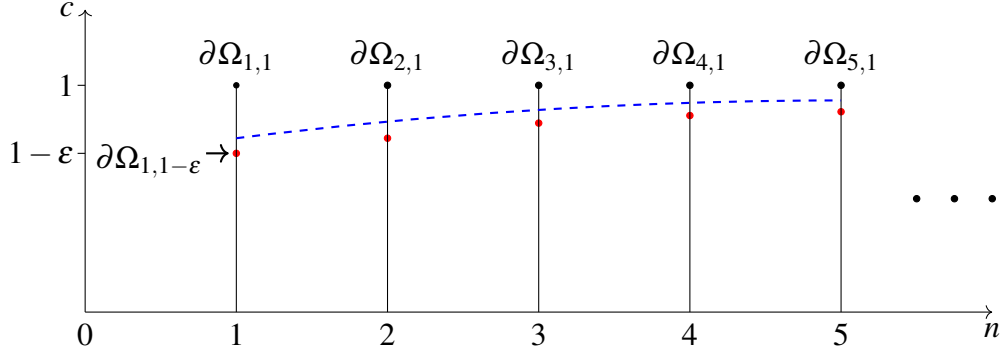


Figure 5.1: The roles of Lemmas 5.3.2 and 5.3.3.

versions of part of Fornæss' proof of theorem 8 in [Fornæss, 2004].

**Proof of lemma 5.3.1.** Let  $\{\varepsilon_n\}, \varepsilon_n \searrow 0$  be given. Using lemma 5.3.2, with  $K_n = \overline{(R\Delta \times R\Delta)}$ , and  $n \in \mathbb{N}$ , we can find  $\{c_n(\varepsilon_n)\}$  and  $\{\hat{\alpha}_n\}$  so that if  $0 < \alpha_n < \hat{\alpha}_n$ , then  $\partial\Omega_{n,c_n} \cap K_{n-1}$  is in an- $\varepsilon$ -tube of  $\partial\Omega_{n,1} \cap K_{n-1}$ .

We can then use lemma 5.3.3 to find sequences  $\{\alpha'_n\}, \{b'_n\}$  so that if  $\alpha'_n = a_1^{b'_n}$ ,  $\alpha_n = a_1^{b_1}$  and  $0 < \alpha_n < \alpha'_n$  and  $b_m \geq b'_m$  and  $b_{m+1} \geq 2b_m$  for  $m \geq n$  and  $\alpha_n$  satisfy the hypotheses of Theorem 8, then  $\Omega_{m,c_n} \subset \Omega$  for all  $m \geq n$ .

Therefore taking  $\alpha_n$  so that  $0 < \alpha_n < \min\{\tilde{\alpha}_n, \hat{\alpha}_n, \alpha'_n\}$ , and ensuring that if for each  $m \geq n$ ,  $\alpha_m = a_1^{b_m}$ , then  $b_m \geq b'_m$  and  $b_{m+1} \geq 2b_m$ , and the  $\alpha_m$  satisfy (5.2.1), the lemma is proven.  $\square$

We illustrate  $c$  as a function of  $n$  in figure 5.1. Given a sequence of decreasing positive numbers  $\{\varepsilon_n\}, 0 < \varepsilon_1 < 1$ , we use lemma 5.3.2 to get the  $c$ -neighborhoods of the  $\Omega_{n,1}$ . In figure 5.1 these neighborhoods are denoted by the vertical line segments with a black dot on the top and a red dot on the bottom. The  $\Omega_{n,c}$  in these neighborhoods are  $\varepsilon_n$ -close to  $\Omega_{n,1}$  on the set  $K_{n-1}$ . Then we use lemma 5.3.3 to pull a sequence of  $\Omega_{n,c_n}$  domains, that will be in the short- $\mathbb{C}^2$   $\Omega$ , into these prescribed neighborhoods. In figure 5.1 these  $\Omega_{n,c_n}$  are denoted by the points of intersection of the dotted blue curve with the black vertical line segments.

Now we will prove lemmas 5.3.2 and 5.3.3. To prove lemma 5.3.2 we will use lemma 5.3.4 and lemma 5.3.5. Lemma 5.3.4 is a generalization of lemma 3.5.1. Lemma 3.5.1 allows us to control level sets of the form  $\{(z, w) \in \mathbb{C}^2 : |z^2 + \alpha w| = 1\}$ . We want to analyze level sets of other heights,

that is, sets the form  $\{(z, w) \in \mathbb{C}^2 : |z^2 + \alpha w| = \beta\}$ , where  $\beta$  need not be equal to one. This is what lemma 5.3.4 does. More explicitly,

$$|\beta z^2 + \alpha w| = 1 \Leftrightarrow \left| z^2 - \frac{\alpha}{\beta} w \right| = \beta^{-1}$$

i.e. On the set  $|w| \leq R$  lemma 3.5.1 describes preimages of level sets of  $|z^2 + \alpha w|$  of height  $\beta^{-1}$  as small  $C^\ell$ -permutations cylinders of radius  $\beta^{-\frac{1}{2}}$ .

The purpose of lemma 5.3.5 is to guarantee the existence an open set of  $\Omega_{n,c}$ 's in a prescribed  $\varepsilon$ -tube of  $\Omega_{n,1}$ .

**Lemma 5.3.4.** *Let  $0 < R < \infty$ . Given  $\varepsilon > 0, \beta > 0, \ell \in \mathbb{N}$  there exists  $\alpha_0 > 0$  such that for every real  $\alpha$ ,  $0 < \alpha < \alpha_0$ , there is a function  $\phi_\alpha \in C^\ell(\partial\Delta \times R\Delta)$  such that*

$$\{(z, w) \in \mathbb{C}^2 : |\beta z^2 + \alpha w| = 1, |w| \leq R\} = \{(\phi_\alpha(\xi, w)\xi, w) : \xi \in \partial\Delta, |w| \leq R\}$$

$$\text{and } \left\| \phi_\alpha - \beta^{-\frac{1}{2}} \right\|_{C^\ell(\partial\Delta \times R\Delta)} < \varepsilon.$$

**Lemma 5.3.5.** *Given  $n \in \mathbb{N}$ ,  $\varepsilon > 0$ , we can find  $\tau = \tau(n) > 0$  such that if  $1 - \tau < c < 1$  then  $\partial\Omega_{n,c}$  is in an  $\varepsilon$ -tube neighborhood of  $\partial\Omega_{n,1}$ .*

We will prove lemma 5.3.5 after proving 5.3.3 and we will prove lemma 5.3.4 at the end of this chapter.

**Proof of lemma 5.3.2.** Suppose we are given  $\varepsilon > 0, n \in \mathbb{N}$  and  $K_n$  a compact subset of  $\mathbb{C}^2$  such that  $\partial\Omega_{n,1} \cap K_n = \{(z, w) : |z_n(z, w)| = 1\} \cap K_n$ . Assume the shears  $F_1, \dots, F_n$  have been chosen and they satisfy Theorem 8. By lemma 5.3.5 we can find  $0 < c_\varepsilon < 1$  such that if  $c_\varepsilon < c < 1$ , then  $\partial\Omega_{n,c} \cap K_n$  is in an  $\varepsilon$ -tube of  $\partial\Omega_{n,1} \cap K_n$ . Call this tube  $T_{n,\varepsilon}$ . Also note that  $\partial\Omega_{n,c} \cap K_n$  is bounded away from the set  $\{(z, w) : |z_n(z, w)| = \varepsilon\}$ . Now use lemma 5.3.4 to find  $\alpha'_{n+1} > 0$  such that if



$0 < \alpha_{n+1} < \alpha'_{n+1}$ , we have that

$$\partial\Omega_{n+1,c_\varepsilon^2} \cap K_n \text{ is uniformly close enough to } \partial\Omega_{n,c_\varepsilon} \cap K_n \text{ that } \partial\Omega_{n+1,c_\varepsilon^2} \cap K_n \subset T_{n,\varepsilon}. \quad (5.3.4)$$

$$\partial\Omega_{n+1,1} \cap K_n \text{ is uniformly close enough to } \partial\Omega_{n,1} \cap K_n \text{ that } \partial\Omega_{n+1,1} \cap K_n \subset T_{n,\varepsilon}. \quad (5.3.5)$$

Strictly speaking, the uniform closeness given by lemma 5.3.4 is in the  $(z_n, w_n)$ -variables. However, if we take an  $\varepsilon$ -tube around  $\partial\Omega_{n+1,1} \cap K_n$ , by the continuity of the map  $F(n)^{-1}$ , this gives an open cover of  $F(n)(\partial\Omega_{n+1,1}) \cap F(n)(K_n)$ , from which we can choose a refinement of this cover composed of open balls, and arguing as we do in lemma 5.3.5 we get (5.3.4) and (5.3.5). Letting  $c_{n+1} = \frac{c_\varepsilon^2 + 1}{2}$  we have:  $\partial\Omega_{n+1,c_{n+1}} \cap K_n \subset T_{n,\varepsilon}$ .  $\square$

**Proof of Lemma 5.3.3.** Suppose  $\varepsilon > 0$ ,  $0 < c < 1$ . Let  $\Delta^2(0, c)$  denote the polydisc in  $\mathbb{C}^2$  of polyradius  $(c, c)$ . Suppose  $(z, w) \in \Delta^2(0, c)$ . Recall that  $F_n(z, w) = (z^2 + \alpha_n w, \alpha_n z)$  and  $\alpha_n = a_n^{2^n}$ . Therefore on  $\Delta^2(0, c)$ ,  $|\alpha_n z| < \alpha_n$  and  $|\alpha_n w| < \alpha_n$  and  $|z^2| < c^2$ . Therefore we obtain  $F_n(\Delta^2(0, c)) \subset \Delta^2(0, c^2 + \alpha_n)$ . Pick  $c' \in (c, 1)$  and for  $\ell \in \mathbb{N}$  let  $c_\ell = c(c')^\ell$ . If  $\ell \geq 0$ , then  $2^\ell \geq \ell + 1$ . Fix  $N \in \mathbb{N}$  and assume that  $\alpha_1, \dots, \alpha_{N-1}$  have been chosen. (In the case  $N = 1$ , no such  $\alpha$  have been chosen.) Write  $\alpha_n = a_1^{b_n}$ , and choose  $b'_N > 2^N$  large enough that for  $b_N \geq b'_N$  and  $\ell \geq 0$ , then we have that

$$(\ell + 1)b_N \log(a_1) < \log(c(1 - c)) + (\ell + 1)\log(c'). \quad (5.3.6)$$

Assume that  $\alpha_n$ , for  $n > N$  are chosen so that they satisfy (5.2.1) and they satisfy the condition:

$$\text{For all } n \geq N, b_{n+1} \geq 2b_n. \quad (5.3.7)$$

Then we have

$$\log(\alpha_{N+\ell}) = \log(a_1^{b_{N+\ell}}) = b_{N+\ell} \log(a_1) \leq 2^\ell b_N \log(a_1) \leq (\ell + 1)b_N \log(a_1)$$

$$\log(a_{N+\ell}) < \log(c(1 - c)) + (\ell + 1)\log(c').$$

Then

$$\begin{aligned}\alpha_{N+\ell} &< c(1-c)(c')^{\ell+1} = c(c')^{\ell+1} - c^2(c')^{\ell+1} < c(c')^{\ell+1} - (c(c')^\ell)^2 \\ (c(c')^\ell)^2 + \alpha_{N+\ell} &< c(c')^{\ell+1}.\end{aligned}$$

Therefore  $F_{N+\ell}(\Delta^2(0, c_\ell)) \subset \Delta^2(0, c_{\ell+1})$ , and taking  $\ell \rightarrow \infty$ , we obtain  $\Delta^2(0, c) \subset \Omega$ .  $\square$

**Proof of lemma 5.3.5.** Let  $\varepsilon > 0, n \in \mathbb{N}$  and define  $P_1$  to be the image of the set  $\partial\Omega_{n,1}$  under the function  $F(n)$ , i.e.  $P_1 = F(n)(\partial\Omega_{n,1})$ . Let  $T_\varepsilon = \bigcup_{(z,w) \in \partial\Omega_{n,1}} \{B_\varepsilon(z, w)\}$ . Since  $F(n)$  is a diffeomorphism, the set  $F(n)(T_\varepsilon)$  is open and contains  $P_1$ . Furthermore, the set  $V_\varepsilon$ , defined by  $V_\varepsilon = \bigcup_{(z,w) \in \partial\Omega_{n,1}} F(n)(B_\varepsilon(z, w))$ , is an open cover of  $P_1$ . Since, for each  $(z, w) \in \partial\Omega_{n,1}$ , the set  $F(n)(B_\varepsilon(z, w))$  is open and contains  $F(n)(z, w)$ , for each  $(z, w) \in \partial\Omega_{n,1}$  we can find an open ball of radius  $r = r(z, w) > 0$ , centered at  $F(n)(z, w)$ , inside  $F(n)(B_\varepsilon(z, w))$ . Let

$$W_\varepsilon = \bigcup_{(z,w) \in \partial\Omega_{n,1}} B_{\frac{r(z,w)}{3}}(F(n)(z, w)).$$

$W_\varepsilon$  is an open cover of  $P_1$ . Since  $P_1$  is compact, we may take a finite subcover of  $W_\varepsilon$ , i.e.

$$\left\{ B_{\frac{r(z_1, w_1)}{3}}(F(n)(z_1, w_1)), B_{\frac{r(z_2, w_2)}{3}}(F(n)(z_2, w_2)), \dots, B_{\frac{r(z_J, w_J)}{3}}(F(n)(z_J, w_J)) \right\}.$$

Let  $\tau = \min_{k \in \{1, 2, \dots, J\}} (r(z_k, w_k))$ . Then if  $(z, w) \in \mathbb{C}^2$  and  $\text{dist}((z, w), P_1) < \tau$ , it follows that  $T_\varepsilon$  contains  $F(n)^{-1}(z, w)$ . Therefore if  $c \in (1 - \tau, 1)$ , then  $\partial\Omega_{n,c} \in T_\varepsilon$ .  $\square$

## 5.4 The Smoothness Argument

Let  $0 < R$  and  $0 < \varepsilon < \frac{1}{2}$  be given. We will construct a Short- $\mathbb{C}^2$  domain  $\Omega$  as a limit of domains  $\Omega_n$ , by choosing coefficients  $a_n$  such that composition of the automorphisms  $F_n : \mathbb{C}^2 \rightarrow \mathbb{C}^2$ , defined by  $F_n : (z, w) \mapsto (z^2 + a_n^{2^n} w, a_n^{2^n} z)$ , will yield domains  $\Omega_n$  with the property that the open set  $\Omega_n \cap \{(z, w) \in \mathbb{C}^2 : |z| < R, |w| < R\}$  is an  $\varepsilon$  small  $C^\ell$ -perturbation of the bidisk  $\{(z, w) \in \mathbb{C}^2 : |z| < 1, |w| < R\}$ .

For any  $\varepsilon > 0$ , define the  $\varepsilon$ -neighborhood of  $\Gamma_r$  to be the set of all  $C^\ell$  graphs over  $\partial\Delta \times R\Delta$  given by functions  $t \in C^\ell(\partial\Delta \times R\Delta)$  such that  $\|r - t\|_{C^\ell(\partial\Delta \times R\Delta)} < \varepsilon$ .

We will show that for  $\Omega = \{(z, w) \in \mathbb{C}^2 : F(n)(z, w) \rightarrow 0\}$ , we have that  $\Omega \cap (R\Delta \times R\Delta) = \lim_{n \rightarrow \infty} (\Omega_n \cap (R\Delta \times R\Delta))$  and we will demonstrate that there exist parameterizing functions  $r_n : \partial\Delta \times R\Delta \rightarrow \mathbb{R}$  such that

$$\partial\Omega_n \cap (R\Delta \times R\Delta) = \{(r_n(\xi, w)\xi, w) : \xi \in \partial\Delta, |w| < R\}$$

where the  $r_n$ s give  $C^\ell$ -graphs over  $\partial\Delta \times R\Delta$  that satisfy the inequalities:

$$\|r_{n+1} - r_n\|_{C^\ell(\partial\Delta \times R\Delta)} < \frac{\varepsilon}{2^{n+3}}$$

Assuming the existence of these functions, the sequence  $r_n$  is Cauchy in the Banach space  $C^\ell(\partial\Delta \times R\Delta)$ , thus the sequence converges in  $C^\ell(\partial\Delta \times R\Delta)$ . Let  $r = \lim_{n \rightarrow \infty} r_n$ .

Assume the inequality

$$\|r_1 - 1\|_{C^\ell(\partial\Delta \times R\Delta)} < \frac{\varepsilon}{2^4}.$$

Then it follows that  $\|r - 1\|_{C^\ell(\partial\Delta \times R\Delta)} < \frac{3\varepsilon}{8}$ .

Therefore,  $\partial\Omega \cap (R\Delta \times R\Delta)$  is  $C^\ell$ -smooth. In fact the boundary is the  $C^\ell$ -graph over  $\partial\Delta \times R\Delta$  given by  $r$ .

## 5.5 Two Smoothness Lemmas

The limit process will use two lemmas, lemma 3.5.2 and lemma 5.3.4. Lemma 5.3.4 is used in the  $n = 1$  step of the limit process argument. Both lemmas are used in the general step.

### Proof of Theorem 10.

Given  $0 < R$ ,  $0 < \varepsilon < \frac{1}{2}$ , we define  $r_0$  to be the graph over  $K_{2R}$  given by the constant function

1. Using lemma 5.3.4 we can choose  $\tilde{\alpha}_1$  between zero and one, small enough that if  $0 < \alpha_1 < \tilde{\alpha}_1$ ,

$\Omega_1 = \{(z, w) \in (2R\Delta \times 2R\Delta) : |z^2 + \alpha| = 1\}$  is given by the  $C^\ell$  graph over  $\partial\Delta \times 2R\Delta$  of a function  $r_1$  with

$$\|r_1 - 1\|_{C^\ell(\partial\Delta \times R\Delta)} < \frac{\varepsilon}{2^4}.$$

Define  $\varepsilon_1 = \frac{\varepsilon}{2^3}$ , and for  $n > 1$ ,  $\varepsilon_n = \frac{\varepsilon}{2^{n+3}}$ . Letting  $\Omega_0 = \{(z, w) \in \mathbb{C}^2 : |z| < 1, |w| < 3R\}$  and  $a_1 = \frac{1}{2}$ , we use Lemma 5.3.1 to get a positive numbers  $b'_1$ ,  $\alpha'_1$ , where  $\alpha'_1 = a_1^{b'_1}$  and a constant  $0 < c_1 < 1$  such that if  $a_n = a_1^{b_n}$  for  $n \in \mathbb{N}$  and  $b_1 \geq b'_1$  and both  $b_{n+1} \geq 2b_n$  for  $n \geq 1$  and (5.2.1) holds, then  $\Omega_{n, c_1} \subset \Omega$  for  $n \geq 1$  and  $\partial\Omega_1 \cap (R\Delta \times R\Delta)$  is in an  $\varepsilon_1$ -tube of  $\partial\Omega_{1, c_1} \cap (R\Delta \times R\Delta)$ . Choose  $\alpha_1 = \min\{\tilde{\alpha}_1, \alpha'_1\}$ . Assume that for  $m = 2, \dots, n$  there exist functions  $r_m$  that are  $C^\ell$  graphs over  $\partial\Delta \times R\Delta$  such that

$$\|r_m - r_{m-1}\|_{C^\ell(\partial\Delta \times R\Delta)} < \frac{\varepsilon}{2^{m+3}}$$

and

$$\|r_m - 1\|_{C^\ell(\partial\Delta \times R\Delta)} < \frac{\varepsilon}{2^{m+3}}.$$

Furthermore, for  $m = 2, \dots, n$ , assume we have positive constants  $c_m, b_m$  such that  $a_m = a_1^{b_m}$ ,  $b_m \geq 2b_{m-1}$ , the  $\alpha_m$  satisfy (5.2.1) and if for  $j \geq m$  and both  $b_{j+1} \geq 2b_j$  and (5.2.1) holds, and  $\Omega_{j, c_m} \subset \Omega$  for  $j \geq m$  and  $\partial\Omega_m \cap (R\Delta \times R\Delta)$  is in an  $\varepsilon_m$ -tube of  $\partial\Omega_{m, c_m} \cap (R\Delta \times R\Delta)$ .

$F(n)$  is an automorphism, so we can find an  $\tilde{R} > 0$  so that  $(2R\Delta \times 2R\Delta) \subset \subset F(n)^{-1}(\tilde{R}\Delta \times \tilde{R}\Delta)$ . Using  $R = \tilde{R}$  in Lemma 3.5.2 also gives a positive constant  $\delta$ , we let  $\varepsilon_{n+1} = \min\left\{\frac{\delta}{2}, \frac{\varepsilon}{2^{n+5}}\right\}$ . Then using lemma 5.3.4 we can choose a constant  $\tilde{\alpha}_{n+1}$  small such that if  $0 < \alpha_{n+1} < \tilde{\alpha}_{n+1}$  then in the variables  $(z_{n+1}, w_{n+1})$ , the cylinder

$$\{(z, w) \in \mathbb{C}^2 : |z^2 + \alpha w| = 1, |w| \leq \tilde{R}\} = \{(\phi_\alpha(\xi, w)\xi, w) : \xi \in \partial\Delta, |w| \leq \tilde{R}\}$$

is a  $C^\ell$ -graph given by a function  $\tilde{r}_{n+1}$  over the cylinder  $\partial\Delta \times \tilde{R}\Delta$  in the variables  $(z_n, w_n)$  with

$$\|\tilde{r}_{n+1} - 1\|_{C^\ell(\partial\Delta \times R\Delta)} < \frac{\varepsilon_{n+1}}{2} \leq \frac{\varepsilon}{2^{n+5}}.$$

Let  $\Gamma_{n+1}$  be the graph given by  $\tilde{r}_{n+1}$ . By lemma 3.5.2,  $F(n)^{-1}(\Gamma_{n+1} \cap (2R\Delta \times 2R\Delta))$  is a  $C^\ell$  graph over  $\partial\Delta \times 2R\Delta$  given by a  $C^\ell$  function we'll call  $r_{n+1}$  with

$$\|r_{n+1} - 1\|_{C^\ell(\partial\Delta \times R\Delta)} < \frac{\varepsilon}{2^{n+4}}.$$

Now use Lemma 5.3.1 to get  $\alpha'_{n+1}, b'_{n+1}$  where  $\alpha'_{n+1} = a_1^{b'_{n+1}}$  and a constant  $0 < c_{n+1} < 1$  such that if  $a_n = a_1^{b_n}$  for  $n \in \mathbb{N}$  and  $b_{n+1} \geq b'_{n+1}$  and both  $b_{m+1} \geq 2b_m$  for  $m \geq n+1$  and (5.2.1) holds, then  $\Omega_{n,c_1} \subset \Omega$  for  $n \geq 1$  and  $\partial\Omega_1 \cap (R\Delta \times R\Delta)$  is in an  $\varepsilon_1$ -tube of  $\partial\Omega_{1,c_1} \cap (R\Delta \times R\Delta)$ . Finally we choose  $\alpha_{n+1} = \min\{\tilde{\alpha}_{n+1}, \alpha'_{n+1}, \alpha_n^{2^n}\}$ .  $\square$

#### Proof of Lemma 5.3.4.

The  $\ell = 1$  case:

$$|\beta z^2 + \alpha w| = 1 \Rightarrow |\beta z^2 + \alpha w|^2 = 1 \Rightarrow (\beta z^2 + \alpha w)(\overline{\beta z^2 + \alpha w}) = 1$$

We let  $z = re^{i\theta}$  and expand left hand side of the last equality above. For convenience we define the terms  $A, B, C$ :

$$\beta^2 r^4 + r^2 \beta \alpha (we^{-2i\theta} + \bar{w}e^{2i\theta}) + \alpha^2 |w|^2 - 1 = Ar^4 + Br^2 + C = 0$$

where  $A = 1$  and

$$B = \beta \alpha (we^{-2i\theta} + \bar{w}e^{2i\theta}), \quad C = \alpha^2 |w|^2 - 1.$$

The quadratic formula gives:

$$r^2 = \frac{-B \pm \sqrt{B^2 - 4\beta^2 C}}{2\beta^2}$$

With  $\alpha$  small, the discriminant is positive and real roots appear. Choose the positive root ( $r \geq 0$ ).

Note that:

$$\begin{aligned}\frac{\partial B}{\partial \theta} &= i2\beta\alpha \left(-we^{-2i\theta} + \bar{w}e^{2i\theta}\right), \\ \frac{\partial B}{\partial w} &= \beta\alpha e^{-2i\theta}, \quad \frac{\partial B}{\partial \bar{w}} = \beta\alpha e^{2i\theta}, \\ \frac{\partial C}{\partial \theta} &= 0, \quad \frac{\partial C}{\partial w} = \alpha^2 \bar{w}, \quad \frac{\partial C}{\partial \bar{w}} = \alpha^2 w,\end{aligned}$$

and therefore

$$\left|\frac{\partial B}{\partial \theta}\right| \leq 4\beta\alpha|w|, \quad \left|\frac{\partial B}{\partial \bar{w}}\right| = \left|\frac{\partial B}{\partial w}\right| \leq \beta\alpha|w|, \quad \left|\frac{\partial C}{\partial \bar{w}}\right| = \left|\frac{\partial C}{\partial w}\right| \leq \alpha^2|w|.$$

Implicitly differentiating (5.5.1)

$$r^2 = \frac{-B + \sqrt{B^2 - 4\beta^2 C}}{2\beta^2} \quad (5.5.1)$$

gives:

$$2r \frac{\partial r}{\partial \theta} = -\frac{\frac{\partial B}{\partial \theta}}{2\beta^2} + \frac{1}{2\beta^2 \sqrt{B^2 - 4\beta^2 C}} B \frac{\partial B}{\partial \theta}$$

and

$$2r \frac{\partial r}{\partial w} = -\frac{\frac{\partial B}{\partial w}}{2\beta^2} + \frac{1}{4\beta^2 \sqrt{B^2 - 4\beta^2 C}} \left(2B \frac{\partial B}{\partial w} - 4\beta \frac{\partial C}{\partial w}\right)$$

If  $|w| \leq R < \infty$ , then one can choose  $\alpha$  small so that

$$\left|\frac{\partial B}{\partial \theta}\right|, \left|\frac{\partial B}{\partial w}\right|, \left|\frac{\partial B}{\partial \bar{w}}\right|, \left|\frac{\partial C}{\partial w}\right|, \left|\frac{\partial C}{\partial \bar{w}}\right|, |B|$$

are small and  $|4\beta C| = \left|4\beta(\alpha^2|w|^2 - 1)\right| \approx 4\beta > 1$ , then the partial derivatives

$$\frac{\partial r}{\partial \theta}, \quad \frac{\partial r}{\partial w}, \quad \frac{\partial r}{\partial \bar{w}}$$

have small modulus. Furthermore, choosing  $\alpha$  small enough so that  $\frac{1}{\beta} |B| < \varepsilon$  and  $\alpha^2 R^2 < \varepsilon$  gives:

$$r^2 \leq \frac{1}{2\beta^2} \left( |B| + \sqrt{|B|^2 + 4\beta^2(\alpha^2 R^2 + 1)} \right) \leq \frac{1}{2\beta^1} (\varepsilon + \sqrt{\varepsilon + 4(\varepsilon + 1)}) \leq \frac{2(1 + 2\varepsilon) + \varepsilon}{2} \beta^{-1}$$

and

$$r^2 \geq \frac{1}{2\beta^2} \left( -|B| + \sqrt{|B|^2 - 4\beta^2(\alpha^2 R^2 - 1)} \right) \geq \frac{1}{2\beta^1} (-\varepsilon + \sqrt{0 + 4(\varepsilon - 1)}) \leq \frac{2(1 - \varepsilon) - \varepsilon}{2} \beta^{-1}$$

Since the constant  $\beta$  factors out of the right-hand side of (5.5.1), the proof of the case  $\ell > 1$  is the same as in the proof of lemma 3.5.1.  $\square$

In this chapter we have constructed a short- $\mathbb{C}^2$  domain with  $C^\ell$ -smooth boundary in a neighborhood of the origin. We guaranteed that  $\Omega$  is a short- $\mathbb{C}^2$  via Fornæss' method of constructing short- $\mathbb{C}^2$  domains (Theorem 8), but were able to control the structure of  $\Omega$  in the polydisc  $R\Delta \times R\Delta$  so that the resulting short- $\mathbb{C}^2$  domain is an arbitrarily small  $C^\ell$  perturbation of the cylinder  $\partial\Delta \times R\Delta$ .

One direction we might strengthen this theorem would be to construct  $\Omega$  so that  $\partial\Omega \cap (R\Delta \times R\Delta) = \partial\Delta \times R\Delta$ . This may be possible but it seems apparent that we would need other tools to prove it. Perhaps the closest such result is in the context of Fatou-Bieberbach domains. In [Wold, 2012] Wold proved that there exists a Fatou-Bieberbach domain  $\Omega$  intersecting the complex plane in the unit disc, though his proof does not guarantee there are not other connected components of  $\Omega$  in the plane.

# Chapter 6

## Short- $\mathbb{C}^2$ with $C^\infty$ -Smooth Boundary

In this chapter we will construct a short- $\mathbb{C}^2$  domain  $\Omega$  with globally  $C^\infty$ -smooth boundary. We will obtain  $\Omega$  as the increasing union of a sequence of approximating sets  $\Omega_n$ . Each  $\Omega_n$  will be the preimage of a polydisc under a composition of  $n$  polynomial shears. The idea of the proof is to choose the domains and the coefficients of the shears together so that  $\Omega_{n-1}$  has a  $C^{n-1}$ -smooth boundary in a polydisc while  $\Omega_n$  will have  $C^n$  smooth boundary in a larger polydisc, and at the same time the shears satisfy the hypotheses of Fornæss's theorem for constructing short  $\mathbb{C}^2$ 's (Theorem 8). The boundary of each set  $\Omega_n$  will locally look like a cylinder, but only in the coordinates given by the composition of the first  $n$  shears. Our main result is:

**Theorem 11 (C).** *There exists a short- $\mathbb{C}^2$  domain  $\Omega$  with  $C^\infty$ -smooth boundary.*

The boundary  $\partial\Omega$  will be constructed as the limit of a sequence of boundaries  $\partial\Omega_j$  restricted to an increasing sequence of subsets  $B_n$  of  $\mathbb{C}^2$  with  $\cup B_n = \mathbb{C}^2$ . Each set  $B_n$  will be an image of a polydisc under an automorphism  $F(n)$ . Define the polynomial shears

$$F_n(z, w) = (z^2 + \alpha_n w, \alpha_n z)$$

Where the constants  $\alpha_n$  will be chosen to satisfy Theorem 8:

$$0 < \alpha_1 < 1, \quad \alpha_{n+1} \leq \alpha_n^{2^n} \text{ for } n \geq 1.$$



Let  $F(n)(z, w) = F_n \circ F_{n-1} \circ \cdots \circ F_1(z, w)$ . We denote the components  $F(n)(z, w)$  by  $(z_n, w_n)$ , i.e.  $F(n)(z, w) = (z_n(z, w), w_n(z, w))$ . For  $0 < c \leq 1$  and  $n \in \mathbb{N}$ , define

$$\Omega_{n,c} = \{(z, w) \in \mathbb{C}^2 : |z_n(z, w)| < c, |w_n(z, w)| < c\}$$

and let  $\Omega_n = \Omega_{n,1}$ . Later we will show that on a sequence of nested compact sets  $B_n$  that union up to  $\mathbb{C}^2$ , for each  $n \in \mathbb{N}$ , the boundaries  $\partial\Omega_m \cap B_n$  converge as  $m \rightarrow \infty$  and so define the boundary of their limit intersected with the set  $B_n$ . Suppose that  $\lim_{n \rightarrow \infty} \Omega_n$  exists and let  $\lim_{n \rightarrow \infty} \Omega_n = \tilde{\Omega}$ . We will demonstrate that  $\tilde{\Omega}$  is a short- $\mathbb{C}^2$  domain using the following argument, which is essentially a variant of Lemma 5.3.1:

We will choose the coefficients  $\alpha_n$  of the shears so that  $\lim_{n \rightarrow \infty} \Omega_{n,1} = \Omega$ , where  $\Omega = \{(z, w) \in \mathbb{C}^2 : F(n)(z, w) \rightarrow 0\}$ . Since  $\Omega_{n,c} \subset \Omega_{n,1}$  for every  $n \in \mathbb{N}$ , we have  $\Omega \subset \lim_{n \rightarrow \infty} \Omega_{n,1}$ . We need to show that  $\Omega \supset \lim_{n \rightarrow \infty} \Omega_{n,1}$ . For any set  $K \subset \mathbb{C}^2$ , and  $\varepsilon > 0$ , we define the  $\varepsilon$ -tube of  $K$  to be the set  $\cup_{(z,w) \in K} B_\varepsilon((z, w))$ . The plan is to choose a sequence of positive real numbers  $\{\varepsilon_n\}$ ,  $\varepsilon_n \searrow 0$ , and nested compact sets  $K_n$  (converging to  $\mathbb{C}^2$ ) and then choose the coefficients  $\alpha_n$  so that  $\partial\Omega_n \cap K_n = \{(z, w) : |z_n(z, w)| = 1\} \cap K_n$  (the existence of such  $K_n$  is given by lemma 5.3.4 and the limit process) and for every integer  $m \geq 1$ :

1.  $(\Omega_{n,1-\varepsilon_m}) \subset \Omega$  for  $n \geq m$ .
2.  $\partial\Omega_{n,1} \cap K_m$  is in an  $\varepsilon_n$ -tube of  $\partial\Omega_{n,1-\varepsilon_m} \cap K_m$  for  $n \geq m+1$ .

If we did this, then given a  $(z, w) \in \text{Int}(\lim_{n \rightarrow \infty} \Omega_{n,1})$ , we can find an  $N$  large enough that  $K_N$  is large enough to contain an open ball centered at  $(z, w)$  that also intersects  $\partial\Omega$ . If  $\delta = \text{dist}((z, w), \partial\Omega \cap K_N)$ , then using 1) and 2) we can find  $M$  large enough that  $\partial\Omega_{M,1-\varepsilon_M} \cap K_N$  is in an  $\delta$ -neighborhood of  $\partial(\lim_{n \rightarrow \infty} \Omega_{n,1}) \cap K_N$ , and so  $(z, w) \in \Omega_{M,1-\varepsilon_M} \subset \Omega$ .

More explicitly, given  $\varepsilon > 0$ ,  $n \in \mathbb{N}$  and  $F_1, \dots, F_n, K_1, \dots, K_n$  we use 2) to get a  $c \in \mathbb{R}$  such that for all choices of  $\alpha_{n+1}$  small enough,  $\partial\Omega_{n+1,c}$  is in an  $\varepsilon$ -tube of  $\partial\Omega \cap K_n$ . Then we use 1) to get upper-bound restrictions on  $\alpha_{n+1}, \alpha_{n+2}, \dots$  so that  $\Omega_{m+1,c} \subset \Omega$  for all integers  $m \geq n$ .

To establish 1), we apply lemma 5.3.3. To establish 2), we use lemma 5.3.2.

Now we will demonstrate boundary convergence and smoothness, and complete the limit process. For  $R_n > 0, \varepsilon > 0$ , let  $A_n$  be the polydisc in the  $(z_n, w_n)$  coordinates given by

$$A_n = A_{n,c,\varepsilon} = \{(z_n, w_n) : |z_n| < R_n, |w_n| < 1 - \varepsilon\}$$

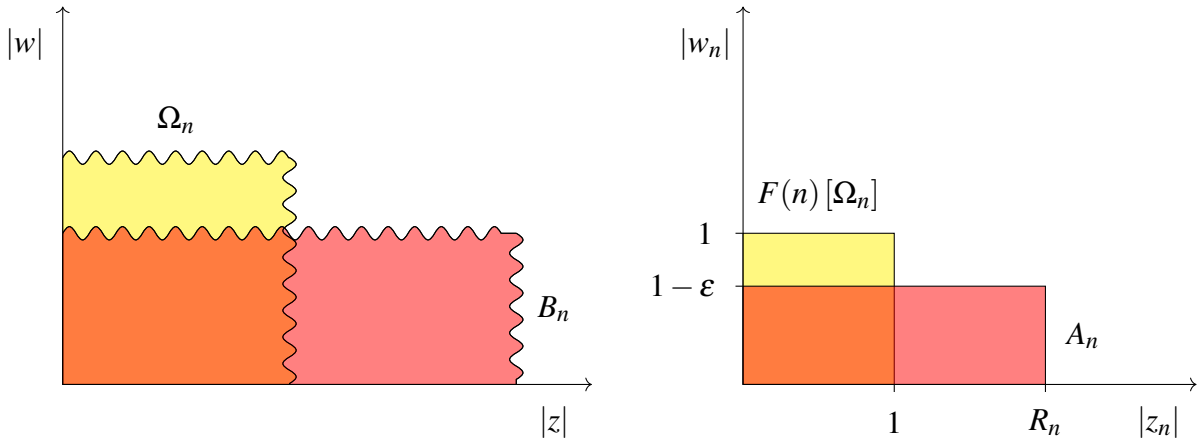
and let  $B_n$  be  $A_n$  in the original  $(z, w)$  coordinates:

$$B_n = B_{n,c,\varepsilon} = \{(z, w) \in \mathbb{C}^2 : |z_n| < R_n, |w_n| < 1 - \varepsilon\}.$$

We will eventually choose a sequence  $\{R_n\}, R_n \rightarrow \infty$ , so that the sets  $B_n$  union up to  $\mathbb{C}^2$ . For  $1 \leq j < n$  we define  $\Omega_n \cap^* A_j$  to mean the set in  $(z_j, w_j)$  coordinates given by

$$\{(z_j, w_j) : (z_j, w_j) = F_{j+1}^{-1} \circ \cdots \circ F_n^{-1}(z_n, w_n) \text{ and } \max\{|z_n|, |w_n|\} < 1\} \cap A_j.$$

The idea is to view the intersection in the coordinates with the smaller of the indices  $n$  or  $j$ . In the same way we define  $\partial\Omega_n \cap^* A_j$ . We will control the boundary of the  $\Omega_j$ 's in the  $B_j$ 's, or the boundaries of the sets  $F(j)[\Omega_j]$  in the sets  $A_n$ .



**The domains  $\Omega_n$  and  $B_n$  and their images under  $F(n)$**

In order to control the boundary of the sets  $\Omega_n, n \geq j$  in the sets  $B_j$  we will construct our domains so that in the set  $B_j$ , in the  $(z_j, w_j)$  coordinates, the domains  $\Omega_n, n \geq j$  are small perturbations

of a cylinder. More explicitly, we want to assume that for each  $n \in \mathbb{N}$  and each  $0 \leq j \leq n$ ,

$$\Omega_n \cap^* A_j = \left\{ (z_j, w_j) : z_j = re^{i\theta}, 0 \leq r < \rho_{j+1}(e^{i\theta}, w_j), 0 \leq \theta < 2\pi, |w_j| < R \right\}$$

and

$$\partial \Omega_n \cap^* A_j = \left\{ (z_j, w_j) : z_j = \rho_{j+1}(e^{i\theta}, w_j)e^{i\theta}, 0 \leq \theta < 2\pi, |w_j| < R \right\}$$

where  $\rho_{j+1}$  is a  $C^n$  function. In order to express this efficiently we will use the following notation:

Let  $n \in \mathbb{N}, 0 < R \in \mathbb{R}$  and define the cylinder in  $(z_n, w_n)$  coordinates

$$P_{n,R} = \{(z_n, w_n) : |z_n| = 1, |w_n| \leq R\}.$$

Let  $r_{n+1} : P_{n,R} \rightarrow \mathbb{R}^+$ . Call the domain

$$\{(z_n, w_n) : z_n = ts, 0 \leq t < r_{n+1}(s, w_n), s \in \partial \Delta, |w_n| < R\}$$

the standard domain over  $P_{n,R}$  given by  $r_{n+1}$ .

Let  $A$  be a subset of  $\mathbb{C}^2$ . By the statement “ $A$  is the  $C^{n+1}$ -graph over  $P_{n,1-\varepsilon}$ , given by the function  $r_{n+1}$ ” we mean that  $r_{n+1} : P_{n,1-\varepsilon} \rightarrow \mathbb{R}_+$  is in the Banach space  $C^{n+1}(P_{n,1-\varepsilon})$  and

$$A = \{(r_{n+1}(\xi, w_n)\xi, w_n) : |\xi| = 1, |w_n| < 1 - \varepsilon\}.$$

We will want to ensure that the sets  $\{B_n\}$  union up to  $\mathbb{C}^2$ . To this end we ensure that these sets contain balls centered at the origin that union up to  $\mathbb{C}^2$ . For  $z \in \mathbb{C}^2$  and  $t > 0$  we denote the Euclidean ball of radius  $t$  and center  $(z_1, z_2)$  by  $B((z_1, z_2), t) = \{(w_1, w_2) \in \mathbb{C}^2 : \|(w_1, w_2) - (z_1, z_2)\| < t\}$ .

**Lemma 6.0.1.** *Suppose that, for each  $n \in \mathbb{N}$  and each  $0 \leq j \leq n$ :*

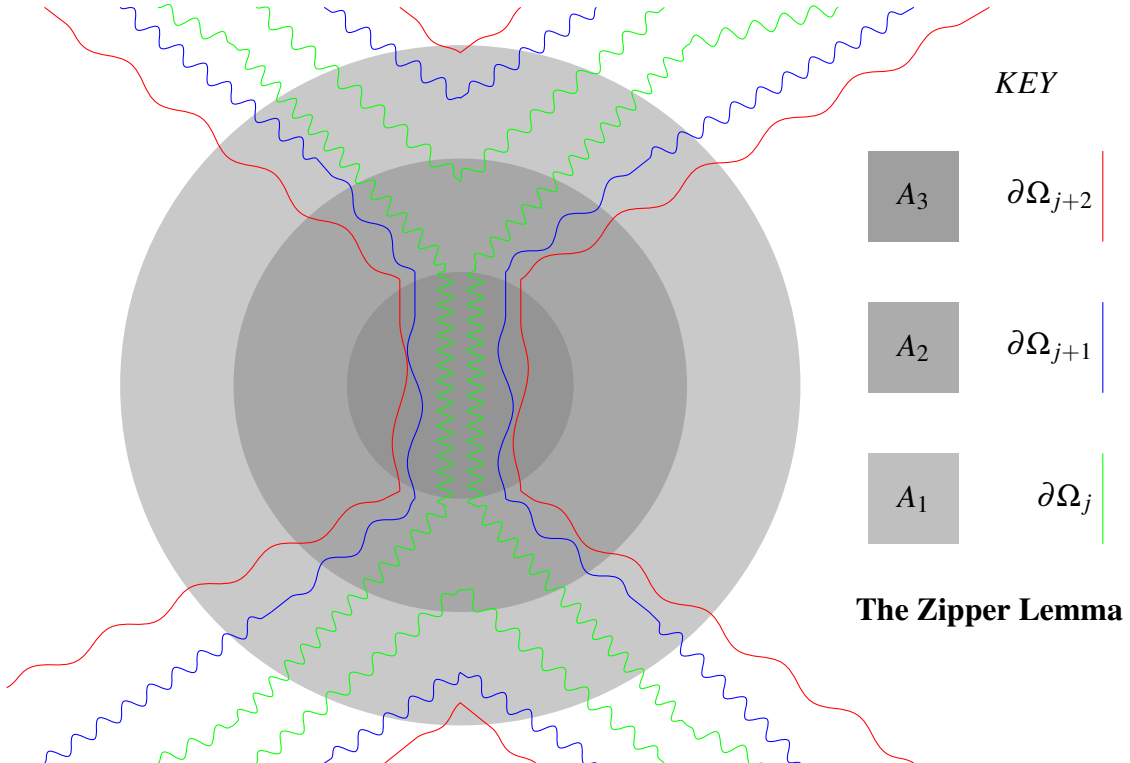
$$1) \quad \Omega_n \cap^* A_j \text{ is a standard domain over } P_{j,1-\varepsilon} \text{ bounded by } \partial\Omega_n \cap^* A_j, \\ \text{a } C^j\text{-smooth graph over } P_{j,1-\varepsilon}. \quad (6.0.1)$$

$$2) \quad \text{The } B_j\text{'s are nested and } B(0, j) \subset B_j. \quad (6.0.2)$$

$$3) \quad \partial\Omega_n \cap^* A_{n-j} \text{ is in an } \frac{\varepsilon}{2^j}\text{-neighborhood of } \partial\Omega_{n-1} \cap^* A_{n-j}. \quad (6.0.3)$$

*Then for any point  $(z, w) \in \partial\Omega$  there exists an  $m_0 = m_0(z, w) \in \mathbb{N}$  such that for any integer  $m \geq m_0$ , the sequence of functions corresponding to the sequence of graphs  $\partial\Omega_m \cap^* A_m, \partial\Omega_{m+1} \cap^* A_m, \partial\Omega_{m+2} \cap^* A_m, \dots$  converges in the Banach space  $C^{m_0}(P_{m_0,1-\varepsilon})$  to a  $C^{m_0}$  function  $\tilde{r}_m$  which gives the graph of the boundary of  $\Omega$  (in the  $(z_m, w_m)$  coordinates) on a set which contains  $(z_m(z, w), w_m(z, w))$ . Hence the map  $F(m)^{-1}$  locally gives a  $C^m$ -smooth boundary approximation of  $\Omega$  containing  $(z, w)$ .*

Looking at (6.0.3) we may think of lemma 6.0.1 as acting like a zipper, where as  $n$  increases we pull up the zipper from the origin, bringing successive domains closer and closer.



*Proof of Lemma 6.0.1.*

If  $(z, w) \in \partial\Omega$  then let  $S = \|(z, w)\|$ , the Euclidean norm of  $(z, w)$ . Then  $(z, w)$  is in all balls  $B(0, m)$  for all integers  $m \geq \lceil S \rceil + 1 = m_0$ . By assumption the graphs  $\partial\Omega_m \cap^* A_m, \partial\Omega_{m+1} \cap^* A_m, \dots$  have restrictions in the Banach space  $C^m(P_{m,1-\varepsilon})$ , and these form a Cauchy sequence in this Banach space, hence they converge to a function  $\tilde{r}_m \in C^m(P_{m,1-\varepsilon})$ .  $\square$

## 6.1 Proof of the Existence of a Short- $\mathbb{C}^2$ Domain $\Omega$ with $C^\infty$ -Smooth Boundary

*Proof of Theorem 11.*

Let  $0 < \varepsilon < \frac{1}{2}$  be given.

For the case  $n = 1$ , using lemma 5.3.2 with  $\varepsilon, n = 0, K_0 = \overline{\Delta \times \Delta}$ , and  $\Omega_{0,1} = \Delta \times \Delta$ , we get  $\hat{\alpha}_1, 0 < c_0 < 1$  such if that  $0 < \alpha_1 < \hat{\alpha}_1$ , then  $\partial\Omega_{1,c_0} \cap K_0$  is in an  $\varepsilon$ -tube of  $\partial\Omega_1 \cap K_0$ .

We use lemma 5.3.4 to choose a positive real number  $\tilde{\alpha}_1, 0 < \tilde{\alpha}_1 < 1$  so that if  $0 < \alpha_1 < \tilde{\alpha}_1$ ,

1)  $\{(z, w) : |z^2 + \alpha_1 w| = 1, |\alpha_1 z| \leq 1 - \varepsilon\}$  is a  $C^1$ -graph given by a function we call  $r_1$  over the cylinder  $P_{0,1}$  and  $|w_1(z, w)| = |\alpha_1 z| < 1 - \varepsilon$  for  $(z, w)$  in the ball  $B(0, 1)$ .

2)  $\|r_1 - 1\|_{C^1(P_{0,1})} < \frac{\varepsilon}{2}$ .

Furthermore, let  $a_1 = \frac{1}{2}$  and we can use lemma 5.3.3 with  $n = 1, \varepsilon$  to find a constant  $\tilde{b}_1 = \tilde{b}_{1,\varepsilon}$  where  $\tilde{b}_1 \geq 2$  and we let  $\alpha'_1 = a_1^{\tilde{b}_1}$ . Let  $\alpha_1 = \min\{\alpha'_1, \tilde{\alpha}_1, \hat{\alpha}_1\}$ . We let  $R_1 = 2$  and define the sets

$$A_0 = \{(z, w) \in \mathbb{C}^2 : |z| < 1, |w| < 1 - \varepsilon\}$$

and

$$A_1 = \{|z_1| < 2, |w_1| < 1 - \varepsilon\}$$

and we define the sets  $B_0 = A_0, B_1 = F_1^{\leftarrow}[A_1]$ . Then we see that

$$B_0 \subset B_1, \quad B(0, 1) \subset B_1.$$

We recall that

$$\Omega_1 = \{(z, w) \in \mathbb{C}^2 : |z_1| < 1, |w_1| < 1\}$$

and therefore  $\partial\Omega_1 \cap B_1$  is the  $C^1$  graph over  $P_{0,1-\varepsilon}$  given by  $r_1$ . Also notice that

$$F_1[\{(z, w) : |z| < 1, |w| < 1\}] \subset \subset \{(z, w) : |z| < 2, |w| < 2\}.$$

Now let  $n \geq 1$  and suppose that for each  $1 \leq j \leq n$ , we have chosen the polynomial shears  $F_j(z, w) = (z^2 + \alpha_j, \alpha_j w)$  where  $\alpha_1^2 \geq \alpha_2$  and for  $2 \leq k \leq n$  the constants  $\alpha_k$  satisfy the condition  $\alpha_k^2 \geq \alpha_{k+1}$ . (We require this to satisfy Theorem 8.)

For  $R > 0$  let  $D_R = \{(z, w) \in \mathbb{C}^2 : |z| < R, |w| < R\}$ . Assume that for each  $1 \leq j \leq n$  there exist constants  $R_j, R_j \geq j$  and the domains  $A_j = \{|z_j| < R_j, |w_j| < 1 - \varepsilon\}$  and  $B_j$ , where

$$B_j = F_1^{\leftarrow} \circ F_2^{\leftarrow} \circ \cdots \circ F_j^{\leftarrow}[A_j],$$

such that

$$\text{i) } A_1 \subset A_2 \subset \cdots \subset A_j \text{ and } B(0, j) \subset B_j.$$

$$\text{ii) } F_j(D_{R_{j-1}}) \subset \subset D_{R_j}.$$

For  $n \in \mathbb{N}, 0 < R \in \mathbb{R}$  we denote the polydisc in the  $(z_n, w_n)$  coordinates of polyradius  $(R, R)$  by  $D_{n,R}$ , that is,  $D_{n,R} = \{(z_n, w_n) : |z_n| < R, |w_n| < R\}$ . Notice that by the previous assumptions, for  $1 \leq j \leq n$ ,  $D_{j-1, R_{j-1}} \subset \subset D_{j, R_j}$ . Also note that by definition,  $A_j \subset D_{j, R_j}$ . Let  $\Gamma_n = \{(z, w) \in \mathbb{C}^2 : |z_n| = 1\}$ .

For  $1 \leq j \leq n$ , let  $\varepsilon_j = \frac{\varepsilon}{2^j}$  and let  $K_j = \overline{B(0, j - \varepsilon)}$ , and assume that  $c_{j, \varepsilon_j}, \alpha_{0,j}$  as in lemma 5.3.2 are determined and that:

1.  $\Gamma_j \cap (R_j \Delta \times R_j \Delta)$  is a  $C^j$ -graph over  $P_{j,R_j}$ , and for,  $1 \leq k \leq j$ :

$$\Gamma_j \cap^* D_{j-k,R_{j-k}} \text{ is in an } \frac{\varepsilon}{2^k}, C^{j-k+1}\text{-neighborhood of } \Gamma_{j-1} \cap^* D_{j-k,R_{j-k}}.$$

2.  $b_{j,c_j}$  are as in lemma 5.3.3 and satisfy  $b_{j+1} \geq 2b_j$  for  $1 \leq j \leq n-1$ .

Then using lemma 3.5.2 (in  $C^1$  through  $C^n$  flavors) to pull  $\partial\Gamma_n$  back to each set of variables  $(z_1, w_1), (z_2, w_2), \dots, (z_n, w_n)$ , restricted to the sets  $D_{1,R_1}, D_{2,R_2}, \dots, D_{n,R_n}$  we find a positive real number  $\delta$  such that if a graph  $\Gamma$  satisfies the condition:

$$\Gamma \text{ is in a } \delta\text{-}C^n \text{ neighborhood of } \Gamma_n \cap D_{n,R_n}.$$

then it follows that, for  $1 \leq j \leq n-1$ :

$$\Gamma \cap^* D_{j,R_j} \text{ is in an } \frac{\varepsilon}{2^{n+1-j}}, C^j\text{-neighborhood of } \Gamma_n \cap^* D_{j,R_j}.$$

Then Use lemma 5.3.4 to choose a constant  $\tilde{\alpha}_{n+1} > 0$  so that if  $0 < \alpha_{n+1} < \tilde{\alpha}_{n+1}$ :

- 1)  $\{(z_n, w_n) : |z_n^2 + \alpha_{n+1} w_n| = 1\}$  is a  $C^n$ -graph given by a function  $r_{n+1}$  over the cylinder  $P_{n,R_n}$  and  $|w_{n+1}(z, w)| = |\alpha_{n+1} z_n(z, w)| < 1 - \varepsilon$  for  $(z, w)$  in the ball  $B(0, n)$  and  $A_n \subset A_{n+1} = \{|z_{n+1}| < R_n, |w_{n+1}| < 1 - \varepsilon\}$ .

- 2)  $\|r_{n+1} - 1\|_{C^{n+1}(P_{n,R_n})} < \delta$ .

We then use lemma 5.3.2 with  $\varepsilon_{n+1} = \frac{\varepsilon}{2^{n+1}}$ , and  $K_n = \overline{(R_n - \frac{\varepsilon}{2^n})\Delta \times (R_n - \frac{\varepsilon}{2^n})\Delta}$  to get  $c_{n+1}$ ,  $\hat{\alpha}_{n+1}$ , such that if  $0 < \alpha_{n+1} < \hat{\alpha}_{n+1}$  then  $\partial\Omega_{n+1,c_{n+1}} \cap K_n$  is in an  $\frac{\varepsilon}{2^n}$ -tube of  $\partial\Omega_{n+1} \cap K_n$ . Then using lemma 5.3.3 with  $c_{n+1}$  we get a constant  $\tilde{b}_{n+1}$ , and we let  $\alpha'_{n+1} = a_1^{\max\{2b_n, \tilde{b}_{n+1}\}}$ .

Finally let  $\alpha_{n+1} = \min\{\hat{\alpha}_{n+1}, \tilde{\alpha}_{n+1}, \alpha'_{n+1}\}$  and choose a constant  $R_{n+1} \geq n+1$  so that

$$F_{n+1}(D_{n,R_n}) \subset\subset D_{n+1,R_{n+1}} \text{ and let } \varepsilon_{n+1} = \frac{\varepsilon}{2^{n+1}}.$$

It follows that for or each  $j \in \mathbb{N}$  there exist constants  $R_j$  and domains  $A_j = \{|z_j| < R_j, |w_j| < 1 - \varepsilon\}$  such that the  $A_j$ 's are nested and  $B(0, j) \subset B_j$ . Furthermore  $\Omega_n \cap^* A_j$  is a standard domain over  $P_{j,1-\varepsilon}$  bounded by  $\Omega_n \cap^* A_j$ , a  $C^j$  smooth graph over  $P_{j,1-\varepsilon}$ . Furthermore, the hypotheses ((6.0.1), (6.0.2)) of lemma 6.0.1 are satisfied. The part of the boundary of  $\Omega$  in  $B_n$  is approximated by  $\partial\Omega_n \cap B_n$ . Since  $\partial\Omega_n \cap B_n$  is the  $C^n$ -graph given by  $r_n : P_{n,1-\varepsilon}$  and for  $m > n$ ,

$$\|r_m - r_{m-1}\|_{C^n(P_{n,1-\varepsilon})} < \frac{\varepsilon}{2^{m+1}}$$

(6.0.3) is satisfied, so on the set  $B_n$ ,  $\Omega$  has at least  $C^n$ -smooth boundary, by lemma 6.0.1. Therefore on  $\mathbb{C}^2$ ,  $\Omega$  has  $C^\infty$ -smooth boundary.

□

**Corollary 1.** *Given  $R > 0$  there exists a short- $\mathbb{C}^2$  domain  $\Omega$  such that  $\partial\Omega \cap (R\Delta \times R\Delta)$  is an arbitrarily small  $C^\infty$ -perturbation of the set  $\partial\Delta \times R\Delta$ .*

Notice that Corollary 1 gives a better smoothness result than Theorem 10 while in Theorem 10 the smoothness estimates are done in the native  $(z, w)$ -variables and are hence in the polydisc  $R\Delta \times R\Delta$ , the boundaries of the  $\Omega_n$  are always described by functions over  $\partial\Delta \times R\Delta$  and thus are easier to work with.

*Remark 7.* We have demonstrated the existence of a short- $\mathbb{C}^2$  domain with  $C^\infty$ -boundary. Naturally we would like to improve this result to a Gevrey-class boundary. In Theorem 7 we were able to achieve a Gevrey-class 4 boundary because the boundary of  $\Omega$  was given explicitly by a defining function whose derivatives we estimated directly. Unfortunately, in Theorem 11 we are describing the boundary implicitly. Even with the appropriate Gevrey estimates on the functions that approximate the boundary, we would need other tools to show that our defining functions for the boundary converge strongly enough to give a Gevrey-class limit. Perhaps the methods Sten-sönes used to prove Theorem 6 could be adapted to the short- $\mathbb{C}^2$  setting to yield a short- $\mathbb{C}^2$  domain with Gevrey-class boundary.



# Chapter 7

## Future Work

One direction for future work would be to try to adapt the methods that in chapter 6 give a globally short- $\mathbb{C}^2$  with globally  $C^\infty$ -smooth boundary to construct a globally  $C^k$ -smoothly bounded Fatou-Bieberbach Domain.

Another direction for future work would be to examine the possible boundary smoothness of Long- $\mathbb{C}^2$  domains that are not biholomorphic to  $\mathbb{C}^2$ . A long- $\mathbb{C}^2$  is a domain  $\Omega$  obtained by taking the increasing union of domains  $\Omega_i$  where each  $\Omega_i$  is biholomorphic to  $\mathbb{C}^2$ . Such domains are an active area of research. For example, in 2005 Wold [Wold, 2005] proved that if each domain  $\Omega_i$  is biholomorphic to  $\mathbb{C}^2$  and Runge, then  $\Omega$  is  $\mathbb{C}^2$ . Later in 2010 Wold [Wold, 2010] proved that there exist Long- $\mathbb{C}^2$  that are Fatou-Bieberbach Domains. In the same paper Wold constructed a long- $\mathbb{C}^2$  which is not Stein and therefore not biholomorphic to  $\mathbb{C}^2$ .

One can construct a long- $\mathbb{C}^2$  as a two-dimensional complex manifold as follows. Begin by taking a countable set of disjoint copies of  $\mathbb{C}^2$ ,  $\sqcup_{i \in \mathbb{N}} X_i$  and a set of injective holomorphic maps  $\phi_i : X_i \rightarrow X_{i+1}, i \in \mathbb{N}$ . Put an equivalence relation on the disjoint union  $\sqcup_{i \in \mathbb{N}} X_i$ . two pairs  $(x, X_i), (y, X_j)$  are equivalent if:

- $i = j$  and  $x = y$
- $i > j$  and  $x = \phi_{i-1} \circ \phi_{i-2} \circ \cdots \circ \phi_j(y)$
- $j > i$  and  $y = \phi_{j-1} \circ \phi_{j-2} \circ \cdots \circ \phi_i(x)$

Call this set of equivalence classes  $\widetilde{X}$ . Define maps  $\psi_i : X_i \rightarrow \widetilde{X}$  by sending  $x \in X_i$  to its equivalence class in  $\widetilde{X}$ . Let  $\widetilde{X}_i = \psi_i(X_i)$ . The maps  $\psi_i^{-1} : \widetilde{X} \rightarrow X_i$  provide local charts. Compatibility of charts can be seen from this commutative diagram:

$$\begin{array}{ccc}
 \widetilde{X}_j & \hookrightarrow & \widetilde{X}_{j+k} \\
 \psi_j^{-1} \downarrow & & \downarrow \psi_{j+k}^{-1} \\
 X_j & \xrightarrow{\phi_{j+k-1} \circ \cdots \circ \phi_j} & X_{j+k}
 \end{array}$$

Here the hooked arrow denoting the map from  $\widetilde{X}_j$  to  $\widetilde{X}_{j+k}$  is the inclusion map. Unfortunately, the method Wold used to create a non-Stein long- $\mathbb{C}^2$  relies on existence theorems for the maps involved, and does not appear amenable to quantitatively analyzing boundary smoothness.

# References

- Bedford, E. (2012). What is... a biholomorphic mapping? *Notices Amer. Math. Soc.*, 59(6), 812–814.
- Bedford, E. & Smillie, J. (1991). Polynomial diffeomorphisms of  $\mathbf{C}^2$ . II. Stable manifolds and recurrence. *J. Amer. Math. Soc.*, 4(4), 657–679.
- Bieberbach, L. (1933). Beispiel zweier ganzer Funktionen zweier komplexer Variablen, welche eine schlichte volumtreue Abbildung des  $\mathbf{R}_4$  auf einen Teil seiner selbst vermitteln.
- Burns, Jr., D. & Shnider, S. (1975). Pseudoconformal geometry of hypersurfaces in  $C^n+^1$ . *Proc. Nat. Acad. Sci. U.S.A.*, 72, 2433–2436.
- Dixon, P. G. & Esterle, J. (1986). Michael’s problem and the Poincaré-Fatou-Bieberbach phenomenon. *Bull. Amer. Math. Soc. (N.S.)*, 15(2), 127–187.
- Fatou, P. (1922). Sur certaines fonctions uniformes de deux variables. *C. R. Acad. Sci. Paris*, 175, 1030–1033.
- Fornæss, J. E. (1979). Plurisubharmonic defining functions. *Pacific J. Math.*, 80(2), 381–388.
- Fornæss, J. E. (2004). Short  $\mathbb{C}^k$ . In *Complex analysis in several variables—Memorial Conference of Kiyoshi Oka’s Centennial Birthday*, volume 42 of *Adv. Stud. Pure Math.* (pp. 95–108). Tokyo: Math. Soc. Japan.
- Fornæss, J. E. & Sibony, N. (1981). Increasing sequences of complex manifolds. *Math. Ann.*, 255(3), 351–360.

- Fornæss, J. E. & Sibony, N. (1991). Random iterations of rational functions. *Ergodic Theory Dynam. Systems*, 11(4), 687–708.
- Fornæss, J. E. & Stensønes, B. (2004). Stable manifolds of holomorphic hyperbolic maps. *Internat. J. Math.*, 15(8), 749–758.
- Forstnerič, F. (2011). *Stein manifolds and holomorphic mappings*, volume 56 of *Ergebnisse der Mathematik und ihrer Grenzgebiete. 3. Folge. A Series of Modern Surveys in Mathematics [Results in Mathematics and Related Areas. 3rd Series. A Series of Modern Surveys in Mathematics]*. Heidelberg: Springer. The homotopy principle in complex analysis.
- Forstnerič, F. (2012). Holomorphic families of long  $\mathbb{C}^2$ 's. *Proc. Amer. Math. Soc.*, 140(7), 2383–2389.
- Forstneric, F., Globevnik, J., & Stensønes, B. (1996). Embedding holomorphic discs through discrete sets. *Math. Ann.*, 305(3), 559–569.
- Globevnik, J. (1997). A bounded domain in  $\mathbb{C}^N$  which embeds holomorphically into  $\mathbb{C}^{N+1}$ . *Ark. Mat.*, 35(2), 313–325.
- Globevnik, J. (1998). On Fatou-Bieberbach domains. *Math. Z.*, 229(1), 91–106.
- Globevnik, J. & Stensønes, B. (1995). Holomorphic embeddings of planar domains into  $\mathbb{C}^2$ . *Math. Ann.*, 303(4), 579–597.
- Johnson, W. P. (2002). The curious history of Faà di Bruno's formula. *Amer. Math. Monthly*, 109(3), 217–234.
- Krantz, S. G. (2008). The Carathéodory and Kobayashi metrics and applications in complex analysis. *Amer. Math. Monthly*, 115(4), 304–329.
- Narasimhan, R. (1960). Imbedding of holomorphically complete complex spaces. *Amer. J. Math.*, 82, 917–934.

- Poincaré, H. (1890). Sur une classe nouvelle de transcendentes uniformes. *Journal de Mathématiques pures et appliquées*, 6, 313–365.
- Poincaré, M. (1907). Les fonctions analytiques de deux variables et la représentation conforme. *Rendiconti del Circolo Matematico di Palermo*, 23(1), 185–220.
- Range, R. M. (1986). *Holomorphic functions and integral representations in several complex variables*, volume 108 of *Graduate Texts in Mathematics*. New York: Springer-Verlag.
- Rosay, J.-P. & Rudin, W. (1988). Holomorphic maps from  $\mathbf{C}^n$  to  $\mathbf{C}^n$ . *Trans. Amer. Math. Soc.*, 310(1), 47–86.
- Shabat, B. V. (1992). *Introduction to complex analysis. Part II*, volume 110 of *Translations of Mathematical Monographs*. Providence, RI: American Mathematical Society. Functions of several variables, Translated from the third (1985) Russian edition by J. S. Joel.
- Stensönes, B. (1997). Fatou-Bieberbach domains with  $C^\infty$ -smooth boundary. *Ann. of Math. (2)*, 145(2), 365–377.
- Ueda, T. (1986). Local structure of analytic transformations of two complex variables. I. *J. Math. Kyoto Univ.*, 26(2), 233–261.
- Wold, E. F. (2005). Fatou-Bieberbach domains. *Internat. J. Math.*, 16(10), 1119–1130.
- Wold, E. F. (2008). A Fatou-Bieberbach domain in  $\mathbb{C}^2$  which is not Runge. *Math. Ann.*, 340(4), 775–780.
- Wold, E. F. (2010). A long  $\mathbb{C}^2$  which is not Stein. *Ark. Mat.*, 48(1), 207–210.
- Wold, E. F. (2012). A Fatou-Bieberbach domain intersecting the plane in the unit disk. *Proc. Amer. Math. Soc.*, 140(12), 4205–4208.
- Yamanaka, T. (1989). A new higher order chain rule and Gevrey class. *Ann. Global Anal. Geom.*, 7(3), 179–203.